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NOTES AND EXAMPLES

— IN —

MECHANICS;

WITH AN APPENDIX ON THE

GRAPHICAL STATICS OF MECHANISM;

— BY —

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Ithaca, N. Y.

ERRATA.

P. 63. In Fig. 76 for "12.2′" read "12.5′."
P. 69. Last line but one, for "4937 lbs." read "4973 lbs."
P. 70. Second line, for "7063 lbs." read "7027 lbs."
P. 124. Third line, for "pin at A," read "pin at B."
Plate V of Appendix; in Fig. 17 [**A**] for "**S**" read "$\mathbf{S}_2$."

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PREFACE.

THE following pages form a companion volume to the writer's *Mechanics of Engineering*, and contain various notes and many practical examples, both algebraic and numerical, serving to illustrate more fully the application of fundamental principles in Mechanics of Solids; together with a few paragraphs relating to the Mechanics of Materials, and an Appendix on the "Graphical Statics of Mechanism."

Advantage has been taken of the use of preliminary impressions in the class-room to make corrections in the electrotype plates; and it is therefore thought that the present complete edition is comparatively free from typographical errors.

In the Appendix are presented many of the problems of Prof. Herrman's "*Zur Graphischen Statik der Maschinengetriebe*" in what seems to the writer a clearer form than in the original (for reasons stated on the first page of Appendix). In this part of the work, the text and diagrams not being adjacent, alternate pages have been left blank in such a way that any diagram and its appropriate text can be kept in view simultaneously.

Besides his indebtedness to Prof. Herrmann's work, the writer would gratefully acknowledge the kindness of the Messrs. Wiley in securing a higher order of excellence in the execution of the diagrams than had at first been contemplated.

In references to the writer's *Mechanics of Engineering* the abbreviation, M. of E., is used.

CORNELL UNIVERSITY,
Ithaca, N. Y., March, 1892.

PREFACE TO SECOND EDITION.

For this second edition the plates of the first have been carefully revised and corrected, (except as indicated in the *errata* on the opposite page) and an entirely new chapter added (Chap. VIII, pp. 119–133), containing various notes and explanations, as also many examples for practice.

Ithaca, January, 1897.

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(See p. 28 of the Appendix for its Table of Contents).

NOTES AND EXAMPLES IN MECHANICS.

CHAPTER I.

DEFINITIONS. PRINCIPLES. CENTRE OF GRAVITY.

1. **Applied Mechanics** is perhaps a more common term for the same thing than "Mechanics of Engineering." "Pure Mechanics" is another name for Analytical Mechanics, which deals with the subject entirely from a mathematical point of view.

2. **Abstract Numbers.**—In experimental investigations in which formulæ are to be deduced, it is best to throw experimental coefficients into the form of abstract numbers, if possible, for these are immediately comparable with those of another experimenter in the same field, if the latter follows the same plan, whether he uses the same units for space, force, and time, or not.

Thus: if the coefficient of friction be defined as the ratio of the friction [force] to the normal pressure [force] producing it, we obtain the same number for it in a definite experiment, whether we express our forces in pounds or in kilograms.

3. **Forces.**—One of the most important things to be acquired in dealing with the practical problems of this study is a proper conception of forces. We do not use the word *force* in any abstract general sense, nor in any popular sense, such as is instanced in the Note of § 15*c*, M. of E. It should always mean the pull, pressure, rub, attraction (or repulsion), of one body upon another, and always implies the existence of a simultaneous, equal, and opposite force exerted by that other body on the first body, i.e., the *reaction;* but this reaction will not come up for consideration in any problem *unless* this "first" body is under treat-

ment as regards the forces acting on *it.* In most problems in Mechanics we have one or more definite rigid bodies under consideration, one at a time, in whose treatment we must form clear conceptions of the forces acting on it; and *these always emanate from other bodies.*

Hence in no case should we call anything a force unless we can conceive of it as capable of measurement by a spring-balance, and *are able to say from what other body it comes.*

For example, a body said to weigh 30 lbs. lies at rest on a smooth level table, which is the only body with which it is in contact. When considered by itself this body is acted on by only two other bodies in a manner which justifies the use of the word *force;* viz., the action of the earth upon it is a vertical downward attraction (force) of 30 lbs.; while the action of the table upon it is an upward pressure (force) of 30 lbs. (We here ignore the atmosphere whose pressures on the body are balanced in every direction.) But suppose the same body and the table with which it is in contact to be allowed to fall, from rest, in a vacuum. The two bodies, during the fall, remain apparently in as close contact as before; but now the upper body is under the action of only *one force*, viz., the downward attraction of the earth, 30 lbs.; and there is no pressure of the upper body against the table, and consequently no pressure of the table against the upper body.

As another instance, an iron rod rests horizontally on two level-faced supports, at its extremities, and bears a load of 60 lbs. in the middle. When this rod is considered "*free*," i.e., when those other bodies which act on it in a "*force*-able" way are supposed removed (their places being for present purposes taken by the respective forces with which they act on the first body), we find it to be under the action of four forces, viz.: a pressure on its middle, vertical and downward, of 60 lbs. from its load; the downward attraction of the earth on it, i.e., its own weight, say 10 lbs. (which is really distributed among all of its particles, but which, so far as the equilibrium, or state of rest or motion, of the body is concerned, is the same as if applied at the centre of gravity, viz., the middle of the rod); and the two upward pressures of the two supports against the ends of the rod, these being

35 lbs. each. If the nature of the investigation requires it, we may go on and consider one of the supports by itself, or "free"; in which case, whatever the actions of other bodies on it may be, that of the rod will be a downward force of 35 lbs., the equal and opposite of the 35 lbs. upward pressure of the support against the rod. These pressures of the two supports against the rod are usually called the "Reactions of the Supports."

As another instance: a ball of 10 lbs. weight hangs at rest by a cord attached to a support above. The cord is of course vertical. This ball is under the action of two forces, viz., a downward attraction of 10 lbs. emanating from the earth, and an upward pull of 10 lbs. emanating from the cord.

A portion of the above cord, taken in the part under tension, is under the action of two forces, thus: the part just above it exerts an upward pull of 10 lbs. upon it, and that below it exerts a downward pull of 10 lbs. upon it. (We here neglect the weight of the portion of cord considered as presumably very small.) In such a case the tension of the cord is said to be 10 lbs. (not 20 lbs.).

Further illustration. Fig. 1 shows a prismatic rod *CB* leaning against the smooth vertical side of a block. Both rest on a rough horizontal plane. The rod is under three forces, viz.: its weight *G* acting vertically downwards through its middle; the pressure of the wall against it, *P* (which, since the wall-surface is perfectly smooth, must be horizontal and points toward the right); and a third force, *Q*, the pressure of the floor against the rod. Since the rod and block are at rest, *P* and *G* intersecting at *A*, it must be that the floor is sufficiently rough to enable the pressure *Q* to deviate from the vertical (that is, from the normal to plane of floor) by as much as the angle *ABV*, *at least;* for, as will be proved later, if three forces act on a body and it remains at rest, the three lines of action must intersect in a common point.

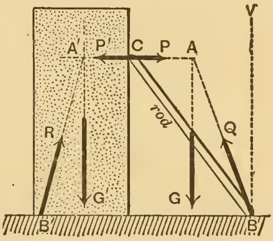


FIG. 1.

We next consider the block, or wall, by itself, and find it to be under the action of P', the equal and opposite of P, and therefore pointing horizontally toward the left; of G', its weight; and a pressure, R, from the floor, whose line of action is determined by the fact that it must be the same line of action as that of the (ideal) resultant of the forces G' and P', since if three forces balance, i.e., are in equilibrium, any one of them must be the equal and opposite of the resultant of the other two and have the same action-line. Forming a parallelogram, therefore, on the forces P' and G', first conceiving each of the two forces to be transferred in its line of action to their point of intersection, A', the diagonal of this parallelogram represents the equal and opposite of R, and has the same line of action.

If this diagonal intersects the floor on the left of the lower left-hand corner of the block, the supposed stability is impossible, unless the block is cemented to the floor.

4. Mass and Weight.—The question of *mass* will be further discussed in a subsequent chapter, p. 53, M. of E. By *weight* we are always to understand the *force* of the attraction which the earth exerts upon the body, and not the amount of matter (mass) in it. This weight will therefore be different in different latitudes and at different distances from the centre of the earth, and requires a *spring-balance* for its determination. Physicists also use the word *weight* in this sense (force).

5. The Heaviness of a body, in the sense used in this study, is something quite different from its total weight. For instance, if the substance of the body is not uniform in composition and density, we cannot speak of the heaviness of the body as a whole, since its various portions have not a common heaviness; however, we may speak of its *average* heaviness, which might be of some use in certain problems, and would be the quotient of its total weight divided by its volume.

Since heaviness is not an abstract number, it would not be sufficient to say that a certain substance has a heaviness of 40, for instance, nor even 40 lbs.; the full statement must be 40 lbs. per cubic foot; which is equivalent to the statement that the heaviness is 0.540 ton per cubic yard.

6. Rigid Body.—As an illustration of the definition in § 10,

M. of E., it may be said that if a horizontal bar, supported at its extremities, is so moderately loaded that the deflection or sinking of the central point is only about one 3-hundredth part, for instance, of the span or distance between the supports, it is sufficiently accurate, for most purposes, to consider that there has been no change in the length of the horizontal projection of any distance measured along the bar. (Where such a consideration is inadmissible, attention will be called to it.)

7. Equilibrium.—Besides speaking of a system of forces being in equilibrium, the phraseology is also sometimes used that the *rigid body is in equilibrium* under the forces acting on it (§ 40a).

The reservation made in § 11, M. of E., as to *state of motion* refers to the fact that any alteration in the distribution of forces acting on a rigid body will usually cause a difference in the internal strains and stresses produced, though the state of motion may or may not be affected, according as any second system of forces applied to the body, on removal of the first, has a different resultant from that of the first, or the same resultant.

8. Division of the Subject.—As to the division given in § 12, M. of E., Sir William Thomson, the noted English physicist, has adopted a different nomenclature, which is getting into wider and wider use. He makes the term *Dynamics* include both statics and dynamics (i.e., what is here and by Rankine and Continental writers called Dynamics), and replaces their word Dynamics by *Kinetics*.

9. Transmissibility of Force. Resultant.—The principle of the transmissibility of force refers only to the state of motion of the rigid body. For instance (see Fig. 2), as far as the *rest* or *motion* of the sickle-shaped body is concerned, it is immaterial whether the force P balance P' (being equal and opposite to it and in the same line) by being applied at O, or by being applied at A; but in the former case the part ABO would be under a bending strain, and in the latter would be under no strain.

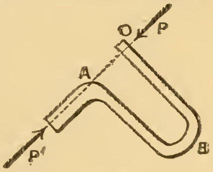


Fig. 2.

It must be remembered that the *resultant* of a given system of forces is always a purely *imaginary* force; that is, all we mean is this: that *if* the given forces were removed and their resultant acted in their stead, *in proper position* as well as magnitude, the state of motion of the rigid body would not be any different from what it would be without the replacement.

10. Parallelogram of Forces; or Triangle of Forces.—By some it is contended that this geometrical construction, or relation, the Parallelogram of Forces, should give place to the *triangle* of forces; i.e., that the resultant (a line laid off to scale representing it) is equal to the third side of a triangle whose other two sides are the two given forces; but a construction of that nature does not show the resultant as acting through the same point as the two components, which should be the case if the construction is to give the position, as well as the magnitude of the resultant.

It is true that the systematic application of the triangle of forces gives rise to the methods of Graphical Statics, as will be seen, but in that case the magnitudes of *all* the forces (or rather lines of definite length representing them and parallel to them) are drawn on a *separate part of the paper* from that containing the figure of the body acted on.

11. General Remarks on Forces.—It is not such a simple matter as it might at first appear, to bring to bear upon a given body a force of prescribed magnitude and direction at a specified point. For example, if we have the idea that a given tension can be produced in a vertical cord by hanging upon it a body whose weight is the given amount, we must remember that the tension in the cord will be equal to the body's weight *only in case the body is at rest or moving in a right line with unchanging velocity* (i.e., describing equal spaces in equal times).

While we can always be sure that the weight of the body itself (or action of the earth on it) is a force of constant value acting on it at all times and in a vertical downward direction, the values of the pressures or pulls exerted upon it by neighboring bodies depend on the *state of motion* of the bodies concerned, as well as on their weights. For instance, suppose that we wish to observe the effect, upon a block of metal placed on a smooth level table and weighing 32.2 lbs., of bringing to bear upon it a horizontal force of 10 lbs. If (see Fig. 3) we attach to it a light flexible (but inextensible) cord, led off horizontally to the right and then passing over a frictionless and very light pulley, *M*, and fasten a 10-lb. weight, *B*, to the end of the right-hand vertical portion. we find, by interposing a light spring-balance between the two parts of the cord at *A*, that when motion is allowed to begin (the cord being pulled straight before the right-hand weight is allowed to sink freely), the tension at *A* is not 10 lbs., but only 7.63 lbs.

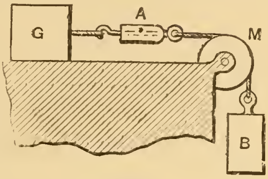


FIG. 3.

If now we gradually increase the weight at *B* in successive experiments until the spring-balance in the cord at *A* shows a reading of just 10 lbs. (the desired force), we find that the weight at *B* has reached a value of 14.5 lbs. (The reasons for this will appear in Ex. 2 of § 57, M. of E.)

The effect of the 10-lb. force on the block is then noted to be as follows: The block begins to move toward the right (having been initially at rest) with an accelerated motion. In the first second it passes over 5 ft.; in the second second, 15 ft.; in the third, 25 ft.; and so on, as the odd numbers, 1, 3, 5, etc. In other words, the *total* distance from the start is equal to 5 *ft.* × *the square of the total time in seconds*.

12. Review of the Fundamental Theorem of the Integral Calculus.—This must be thoroughly reviewed and understood before using the integral calculus in the theory of the centre of gravity, or elsewhere in Mechanics. The problems to which we apply the integral calculus will almost always be of a nature calling for the summing up of a vast number of very small quantities all of which are alike in form and usually consist of the product of two or more factors, one of which [the *differential*] is very small, from which it comes that the product itself is small. (This is the most available form in which to conceive of the operation, but, as will be seen, it is

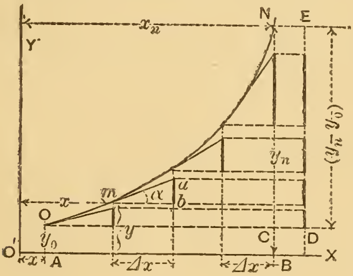


FIG. 4.

after all a fictitious description of what is done, being put into that shape to avoid the continued repetition of a very lengthy phraseology.) This theorem and operation are most readily appreciated by the aid of graphic representation, as follows:

Let ON be any portion of any algebraic curve, its equation being

$$y = func.\text{ of } x; \quad \text{or,} \quad y = \phi(x). \quad . \quad . \quad . \quad . \quad (1)$$

At any point m of the curve there is a special value of x and of y, and also of the angle α, which the tangent-line there drawn makes with the axis X.

From the Differential Calculus we know that tan α is the "first differential coefficient" of y with respect to x (or "first derivative," or simply "derivative"), and may be written $\phi'(x)$; whence with symbols,

$$\tan \alpha = \frac{dy}{dx}; \quad \text{or} \quad \tan \alpha = \phi'(x). \quad . \quad . \quad . \quad . \quad (2)$$

Now divide the distance $A \ldots B$ into a large number of parts, not necessarily equal, the length of any one being called Δx, and raise an ordinate at each point of subdivision. Where each such ordinate cuts the curve, as at m, draw a tangent line to the curve, and a horizontal line as $m \ldots b$. These two lines intercept a small length $a \ldots b$ on the next ordinate on the right. All such intercepts are shown drawn in heavy lines. ab is typical of any one of these intercepts. From the right triangle concerned we now have

$$\overline{ab} = \overline{mb} \,.\, \tan \alpha; \text{ i.e., } \overline{ab} = (\tan \alpha)\Delta x = \left[\phi'(x)\right]\Delta x. \quad . \quad (3)$$

Project all the lengths like ab on a convenient vertical line, as ED, and note that their sum is, of course, not quite equal to ED, or NC, which is $y_n - y_0$. This sum we may express as $\Sigma_0^N\left[\phi'(x)\right]\Delta x$; and hence state that

$$\Sigma_0^N\left[\phi'(x)\right]\Delta x \textit{ is almost equal to } y_n - y_0. \quad . \quad . \quad (4)$$

Or, since $y_n - y_0$ may be written $[\phi(x)]_{x=x_n} - [\phi(x)]_{x=x_0}$, $= \Big[_0^n \phi(x)$, we may re-state the fact thus:

$$\Sigma_0^N [\phi'(x)] \varDelta x \textit{ almost} = \Big[_{x_0}^{x_n} \phi(x). \quad . \quad . \quad . \quad . \quad (5)$$

Since it is evidently a geometrical possibility to increase the number of $\varDelta x$'s between A and B until the discrepancy between $\Sigma_0^N [\phi'(x)] \varDelta x$ and $\Big[_{x_0}^{x_n} \phi(x)$ is less than any value that may be assigned, we may say that $\Big[_{x_0}^{x_n} \phi(x)$ is the *limit* which $\Sigma_0^N [\phi'(x)] \varDelta x$ approaches as the subdivision becomes finer and finer. This limit may also be expressed by the notation $\int_{x_0}^{x_n} [\phi'(x)] dx$, and we accordingly write

$$\Sigma_0^N [\phi'(x)] \varDelta x \left\{ \begin{matrix} \textit{approaches a} \\ \textit{limit called} \end{matrix} \right\} \int_{x_0}^{x_n} [\phi'(x)] dx,$$

$$\textit{which limit} = \Big[_{x_0}^{x_n} \phi(x). \quad . \quad (6)$$

The *utility* of all this is that if in any problem we note that the sum of a series of terms, each one of which is the product of a small portion, $\varDelta x$, of the axis of X by a function of x (the same function in all) ($\varDelta x$ being the difference between the value of x for any term and that for the next term) is an approximation to the desired result; and if, furthermore, the desired result is the limit approached in value by the sum of the series as the subdivision becomes finer and finer along the axis of X, then the desired result will be obtained as follows: *Find the anti-derivative of the above function* of x, and substitute in it first, for x, the value, x_n, which x assumes at the upper end of the series; and secondly, substitute the value x_0 of x at the lower end of the series. The difference of the expressions so obtained is the desired result. (NOTE.—By *anti-derivative* we are to understand that function of x, the differentiation of which will give rise to the given function. Thus, the anti-derivative of $\phi'(x) = x^2$ is $\phi(x) = \frac{1}{3}x^3 + \text{const.}$; and in applying the above rule we may omit the constant, knowing that it would cancel out in the subtraction indicated.)

13. Example of the Foregoing.—We wish to find by calculus the total *moment* of the homogeneous paraboloid of revolution in Fig. 5 about the axis Y; the axis of the solid being horizontal and coinciding with the axis X. (By "moment" of a small weight we mean the product of the weight by the length of the perpendicular let fall upon it from a given line or plane). The solid is bounded on the right by a circular base, KN, perpendicular to axis X. Conceiving the solid to be divided into a great number of circular disks, or laminæ, perpendicular to axis X, all of the same thickness, Δx, but of different radii, u, and denoting the "heaviness" of the substance by γ, we have $\gamma\pi u^2\Delta x$ as the weight of any disk, and $\gamma\pi u^2\Delta x\,[x+\frac{1}{2}\Delta x]$ as its moment about the axis Y. (The axis Y is perpendicular to the paper through O.) The equation to the parabola being $u^2 = 2px$, where p is the *parameter*, this moment may also be written $2\gamma\pi px\Delta x[x+\frac{1}{2}\Delta x]$, and hence the sum of all such moments for all the disks, or the *total moment*, may be expressed as

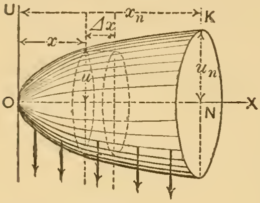


Fig. 5.

$$M' = 2\gamma\pi p\Sigma_0^N[(x^2)]\Delta x + \gamma\pi p\Delta x\Sigma_0^N[(x)]\Delta x. \quad . \quad . \quad (7)$$

This sum does not apply to the paraboloid, but only to the *stepped solid* (or aggregation of disks with square edges), unless Δx is finally made zero. That is, the desired result (moment of actual paraboloid) will be obtained as the limit which M' approaches as Δx is made smaller and smaller. When Δx is made zero, the first term of M' becomes $2\gamma\pi p\int_{x_0}^{x_n}(x^2)dx$; (i.e., $\phi'(x) = x^2$); while as to the second, although the limit of $\Sigma_0^N(x)\Delta x$ is $\int_{x_0}^{x_n}(x)dx$, and hence is not zero, yet the outside factor, Δx, *is* zero and hence the second term vanishes. Therefore the total moment of the paraboloid is $M = 2\gamma\pi p\int_{x_0}^{x_n}[x^2]dx$; and in this we note that the $\phi'(x)$ of the general form of eq. (6) is x^2, and that the x anti-derivative of x^2 is $\frac{1}{3}x^3 +$ const.; and hence, finally,

$$M = 2\gamma\pi p\left(\Big[^{x_n}_{x_0}\left(\tfrac{1}{3}x^3 + C\right)\right) = 2\gamma\pi p\left[\tfrac{1}{3}{x_n}^3 - \frac{0^3}{3}\right] = \tfrac{2}{3}\gamma\pi p {x_n}^3.$$

This quantity divided by the total weight $(= \gamma\pi p {x_n}^2)$ of the solid will give the distance of its centre of gravity from the vertex, or origin, O; i.e., $\bar{x} = \frac{2}{3}x_n$.

As to notation, it is customary to anticipate the fact that the desired result justifies the use of the notation $\int_{x_0}^{x_n}[\phi'(x)]dx$, and to employ at once dx for Δx in making out the form of one term of the series. dx is then called an *infinitesimal*, which simply means that Δx is finally to become zero; in other words, that the result sought is the limit which the sum of the series approaches as Δx diminishes.

14. Position of Centre of Gravity of Various Geometrical Forms. (Homogeneous, etc.; the plane figures representing thin plates of uniform thickness.)

Obelisk. Fig. 6 shows a homogeneous *obelisk*, or solid bounded by six plane faces, of which two are rectangular (horizontal in this figure) with corresponding edges parallel (and hence these rectangular faces are parallel, and may be considered as bases). Required the distance, $\bar{z}$, of the centre of gravity, C', of the obelisk, from the base $EHGF$. See Fig. 6 for notation. h is the perpendicular distance between the bases.

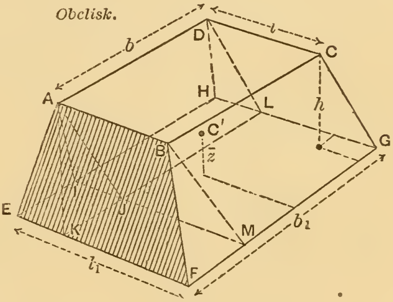


FIG. 6.

By passing a plane through the edge AD || to face $BCGF$; a plane through AB || to $DCGH$; and noting their intersections (dotted lines) with the faces of the obelisk, we subdivide it into the following geometric forms:

A *parallelopiped* $ABCD$-$JMGL$, of volume $V_1 = blh$ and having its centre of gravity at a distance $\bar{z}_1 = \frac{1}{2}h$ above base HF;

A *triangular prism* AD-$IJLH$, of volume $= V_2 = \frac{1}{2}b(l_1 - l)h$ and for whose centre of gravity $\bar{z}_2 = \frac{1}{3}h$;

Another *triangular prism* AB-$KFMJ$, of volume $= V_3 = \frac{1}{2}l(b_1 - b)h$ and for whose centre of gravity $\bar{z}_3 = \frac{1}{3}h$; and finally,

A *pyramid* A-$EKJI$, whose volume is $V_4 = \frac{1}{3}h(b_1 - b)(l_1 - l)$. and whose "mean z" is $\bar{z}_4 = \frac{1}{4}h$.

Hence by eq. (3) of p. 19, M. of E., with $\bar{z}_1$, etc., instead of $\bar{x}_1$, etc., we have, after reduction,

$$\bar{z} = \frac{b_1l_1 + 3bl + b_1l + bl_1}{2b_1l_1 + 2bl + b_1l + bl_1} \cdot \frac{h}{2}.$$

Triangular Plate. Fig. 7. Bisect AB in M. Join OM. Bisect OB in N and join AN. The intersection, C, is the centre of gravity of the triangular plate or plane figure. [The centre of gravity of the mere *perimeter* of the triangle (slender wires, homogeneous and of same sectional area) is the centre of the circle inscribed in a triangle formed by joining the middles of the sides.]

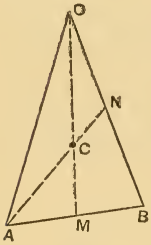


Fig. 7.

Parabolic Plate. Fig. 8. AN being ┐ to axis ON. $\bar{x} = \frac{3}{5}\overline{ON}$.

Upper Half of preceding Parabolic Plate. Fig. 9. $\bar{x} = \frac{3}{5}\overline{ON}$ and $\bar{y} = \frac{3}{8}\overline{AN}$.

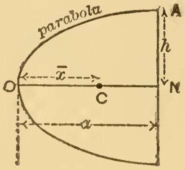


Fig. 8.

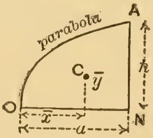


Fig. 9.

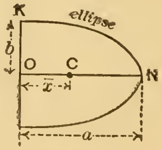


Fig. 10.

Semi-ellipse. Fig. 10. Semi-axes are a and b; $\bar{x} = \dfrac{4a}{3\pi}$.

Sector of a Sphere. $AEBO$, Fig. 11. Let $h =$ the altitude of the zone, or cap, of the sector; i.e., let $h = r - \overline{OK}$; then $\overline{OC} = \frac{3}{4}r - \frac{3}{8}h$; C being the centre of gravity.

Segment of a Sphere. AK, Fig. 12. Let the altitude, AK, of the segment be h, and C the centre of gravity; then

$$\overline{OC} = \frac{3}{4} \cdot \frac{(2r-h)^2}{3r-h}.$$

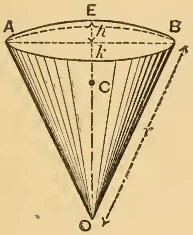


FIG. 11.

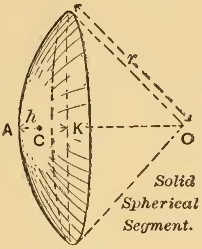


FIG. 12.

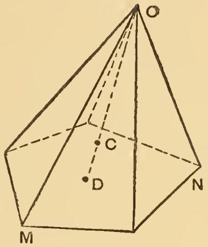


FIG. 13.

Any Pyramid (or Cone). Fig. 13. Join the vertex O with D, the centre of gravity of the base MN. From D lay off $\overline{CD} = \frac{1}{4}\overline{DO}$. C is the centre of gravity of the solid.

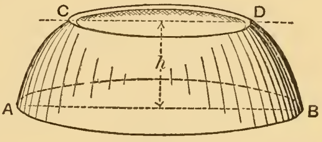


FIG. 14.

Zone on Surface of Sphere. Fig. 14. (Thin shell, homogeneous and of uniform thickness.) The centre of gravity lies at the middle of the altitude, h, in the axis of symmetry. The small circles of the sphere, CD and AB, lie in parallel planes.

15. Simpson's Rule (Fig. 15).—If $ABCDEFG$ is a smooth curve and ordinates be drawn from its extremities A and G to the axis X, an approximation to the value of the area so enclosed, $A..D..G..N..O..A$, between the curve and the axis X, is obtained by *Simpson's Rule*, now to be demonstrated. Divide the base ON into an even number, n, of equal parts, each $= \Delta x$ (so that $\overline{ON} = n \, . \, \Delta x$), and draw an ordinate from each point of division to the curve, the lengths of these ordinates being u_1, u_2, etc.; see figure.

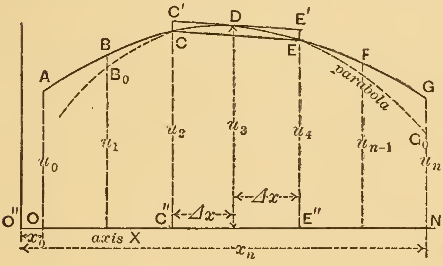


FIG. 15.

Consider the strips of area so formed in consecutive pairs; for example, *CDEE''C''* is the second pair in this figure (counting from left to right). Conceive a parabola, with its axis vertical, to be passed through the points *C*, *D*, and *E*. It will coincide with the real curve between *C* and *E* much more closely than would the straight chords *CD* and *DE*; and the segment *CDE*, considered as the segment of this parabola, has an area equal to two thirds of that of the circumscribing parallelogram *CC'E'E*. Hence, since the area of this pair of strips = *trapezoid CEE''C''* + *parabolic segment CDE*, we may put

$$\left.\begin{matrix}\textit{Area of pair}\\ \textit{of strips } CE''\end{matrix}\right\} = 2\Delta x\left[\tfrac{1}{2}(u_2+u_4)+\tfrac{2}{3}(u_3-\tfrac{1}{2}[u_2+u_4])\right];$$

which reduces to $\quad \ldots \tfrac{1}{3}\Delta x[u_2+4u_3+u_4].$

Treating all the $\frac{n}{2}$ pairs of strips in a similar manner, we have finally, after writing $\Delta x = (x_n - x_0) \div n$,

$$\left.\begin{matrix}\textit{Whole area}\\ AG''\ (\textit{approx.})\end{matrix}\right\} = \frac{x_n - x_0}{3n}\left[u_0+4u_1+2u_2+4u_3+2u_4+\ldots+u_n\right].$$

The approximation is closer the more numerous the strips and the more accurate the measurement of the ordinates u_0, u_1, u_2, etc.

If the subdivision on the axis *X* were "infinitely small," an exact value for the area would be expressed by the calculus form $\int_{x=x_0}^{x=x_n} u\,dx$. Hence for any integral of this form, $\int_{x=x_0}^{x=x_n} u\,dx$, if we are only able to determine the particular values (u_0, u_1, etc.) of the variable u corresponding respectively to the abscissæ x_0, $x_0+\Delta x$, $x_0+2\Delta x$, etc. (where $\Delta x = (x_n - x_0) \div n$, n being an *even* number), we can obtain an approximate value of the integral or summation by writing

$$\int_{x_0}^{x_n} u\,dx = \frac{x_n - x_0}{3n}\left[u_0 + 4(u_1 + u_3 + \ldots + u_{n-1}) \right.$$
$$\left. + 2(u_2 + u_4 + \ldots + u_{n-2}) + u_n\right].$$

As to the meaning of n, note that the first ordinate on the left is not u_1, but u_0; also that while there are n strips, the number of points of division is $n+1$, counting the extremities *O* and *N*.

CHAPTER II.

PRINCIPLES AND PROBLEMS INVOLVING NON-CONCURRENT FORCES IN A PLANE.

16. Most Convenient Form for Analytical Conditions of Equilibrium (Fig. 16).—Let P_1, P_2, P_3, etc., constitute a system of non-concurrent forces in a plane acting on a rigid body and in equilibrium. Of the actual system, P_1 and P_2 are the only forces shown in the figure. Assuming a convenient origin, O, introduce into the system two opposite and *equal* forces, P_1' and P_1'', both acting at O and equal and parallel to P_1. Evidently the presence of these two forces does not destroy the equilibrium of the original system. Similarly, introduce at O the mutually annulling forces P_2' and P_2'' bearing the same relation to P_2 (parallelism and equality) that P_1' and P_1'' do to P_1; and so on for each of the remaining forces of the system. Drop a perpendicular from O on each of the forces of the original system, the lengths of these perpendiculars being a_1, a_2, a_3, etc. (a_1 and a_2 are shown in the figure). We now note that for each force P of the original system we have in the new system a single force at O, equal and parallel to P and similarly directed, and also a couple, of moment Pa. For example, the force P_2 of the original system is now replaced by the force P_2'' parallel and equal to P_2 and similarly directed, but *acting at the point O*; and by the couple formed of the two forces P_2 and P_2', the arm of this couple being a_2. It follows, therefore, that the new system consists of a set of forces (P_1'', P_2'', P_3'', etc.), all meeting at O (*and hence forming a concurrent system in a plane*), and a set of couples, of moments P_1a_1, P_2a_2, etc. Since no single force can balance a couple (§ 29, M. of E.) or set of couples, the forces of the concurrent system at O must be in equilibrium among themselves; i.e., X and Y being any two directions at right angles, we must have ΣX

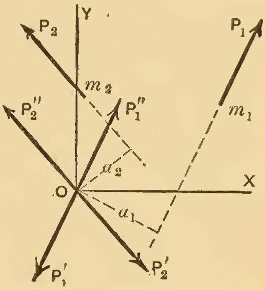


FIG. 16.

and ΣY separately equal to zero for the concurrent system at O; and the set of couples must be in equilibrium among themselves, whence it follows that the *moment of their resultant couple must equal zero.* Since the couples are in the same plane, the moment of their resultant couple is the algebraic sum (see § 34, M. of E.), i.e., $P_1a_1 + P_2a_2 + \ldots$, or $\Sigma(Pa)$, $= 0$.

For the equilibrium, therefore, of a system of non-concurrent forces in a plane, we must have not only ΣX *and* $\Sigma Y = 0$, *but also* $\Sigma(Pa) = 0$. That is, in practical language (the body being originally at rest), the forces of the system so neutralize each other that they not only do not tend to move the body sideways or vertically, *but also do not produce rotation.*

In the practical application of these conditions of equilibrium in solving problems it is not necessary to introduce the pairs of equal and opposite forces at O (the conception of which is needed only for purposes of proof), since the sum of the X-components (or Y-components) of the actual forces of the system is equal to that of the X-components of the auxiliary forces introduced at O; while to form the moment-sum of the auxiliary couples, we have only to multiply each force of the actual system by the perpendicular distance of its line of action from the origin, whose position is taken at convenience.

The student should now read the latter half of p. 33, M. of E.

17. The Rigid Bodies Dealt with at Present.—Each rigid body now to be considered is one whose dimensions perpendicular to the paper are supposed to be very small, and therefore may be considered to lie in the plane of the paper. An actual structure is made up of such pieces, or *members*, which are provided with forked joints, or duplicated in such a way that the above supposition (each piece lying in the plane of the paper) is practically justified.

All surfaces of contact between any two contiguous pieces are supposed perpendicular to the paper, and friction between two such parts of a structure is disregarded; i.e., *the pressure between two contiguous pieces* (or " members") *is in the plane of the paper and normal to the surfaces of contact;* for it is a matter of common experience that pressure can be exerted at the *smooth* surfaces of contact of two bodies *only in a direction normal to those surfaces.*

In problems where the *weights* of one or more bodies connected with the structure are considered, the plane of the structure will be vertical, and then (considering what has already been postulated) the system of forces acting on each member, or piece of the structure, is *a system of forces in a plane* (a "uniplanar" system of forces).

18. Contact Forces or Pressures.—If one of the two bodies in contact is rounded at the point of contact, while the other is quite flat at that point, the action-line of their mutual pressure necessarily lies in a *perpendicular*, or *normal*, to the latter flat surface, and passes through the point of contact. Hence, the shapes of the bodies being known, this action-line becomes known on inspection. Let this be called "*flat-contact pressure.*" (N.B. As a better definition of flat-contact pressure, we might describe it as a pressure occurring in such a way that *its action-line cannot be materially changed by any slight motion of one piece relatively to the other during the small alteration of form and position which actually takes place when a load is gradually placed on the structure.*)

But if the mode of connection of the two bodies is a *pin-connection*, that is, if one body carries a round pin or bolt fitting (somewhat loosely) in a corresponding *ring* forming part of the other body, we are unable to say in advance just where, on the inner circumference of the ring, the contact (and accompanying pressure) is going to be. Wherever the point is, all that we can immediately say as to the action-line of the force is that *it passes through the centre of the circle*, its direction (if determinate at all) being found from a consideration of the other forces acting on the piece in question. This will be illustrated later.

Such a pressure may be called a *hinge-pressure.*

19. Classification of Rigid Bodies under Uniplanar Systems of Forces.—As conducive to clearness in subsequent matter, the rigid bodies composing a structure will be named according to the number of forces acting on each; and these forces consist of gravity actions (i.e., weights) and of the pressures exerted by neighboring pieces on the piece in question. The resultant gravity action on a single piece, or member of the structure, is a single force, called its weight, acting vertically downward through its centre of gravity. (Of course, the action of gravity is distributed over all the particles of a body, but the above-mentioned single force is the full equivalent of these distributed forces *as far as the equilibrium of the piece is concerned; though not such* as regards the *straining action on the piece.* With these straining actions we cannot deal here; they will be treated later in the proper place. In many cases the whole weight of a piece is so small in comparison with any of the other forces of the system acting on that piece that no appreciable error is made in regarding it as without weight. Notice is always given in such cases.)

20. "Two-force Pieces" and their Treatment.—A "two-force" piece being a piece on which only two forces act, if the weight of the piece is considered there is but one other force. For example, Fig. 17, a body of weight G hangs by a stem and ring which form a rigid part of it, on a pin projecting from a fixed support.

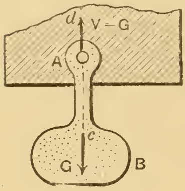


Fig. 17.

Since, evidently, the *equilibrium of a two-force piece requires that the two forces shall be equal and opposite and act in the same line*, the piece $A\,..\,B$ will not be in equilibrium unless the centre of gravity c lies in a vertical line drawn through the centre of the circular section of the pin at A. Here we have an instance of the final full determination (by the necessity of its being vertical) of the action-line of a hinge-pressure, concerning which we know in advance, only that it passes through the centre of the circle at A. We find, therefore, that the pressure of the pin at A against the ring of the rigid body $A\,..\,B$, hanging at rest, must have a direction vertically upward, and an amount, V, numerically equal to G, while its action-line, $c\,..\,d$, passes vertically through the centre of the hinge-circle.

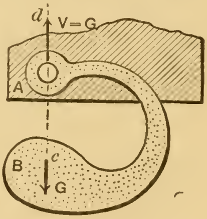


Fig. 18.

In Fig. 18 is another two-force piece, in which, for equilibrium, the same result is reached, but, the stem being curved, the straining action in it is of a bending nature; whereas, in Fig. 17 it is a simple tension, or stretching action.

The *case of a two-force piece whose weight is neglected* is instanced in Fig. 19, where the two-force piece $A \,..\, B$ is subjected to tension through the action of a suspended weight, G, and a bell-crank lever, BCD. The hinge-pressure at A, from the support S against the piece $A \,..\, B$, must pass through the centre of the circle at A; while also the hinge-pressure at B, of the bell-crank against the piece $A \,..\, B$, must pass through the centre of the hinge-circle at B. But these two hinge-pressures are the only forces acting on the piece $A \,..\, B$, and for the equilibrium of a two-force piece must be equal, opposite, and coincident as to action-line; that is, these pressures must both act in the line joining the centres of the two circles. $A'B'$ shows this piece represented as a free body, with the equal and directly opposite forces P' and P' acting at the ends. As to the value, or *amount*, of this force P', it cannot be found until the case of the bell-crank has been treated, depending, as it does, not only upon the design of the bell-crank and the amount of the load G, but also upon the position of the piece AB itself.

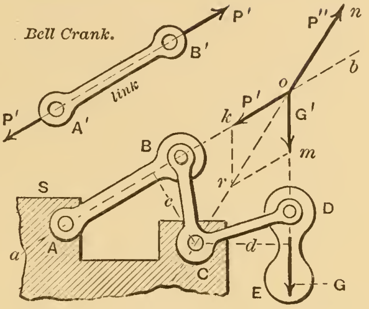


FIG. 19.

21. Three-force Pieces.—If a rigid body is at rest (i.e., in equilibrium) under the actions of three forces, it is evident that these forces must have action-lines intersecting in a common point, and that each force must be the equal and opposite of the diagonal of the parallelogram formed on the other two as sides (laid off to scale); for any one of them *must be the anti-resultant of the other two.* (See § 15. M. of E.) [In the particular case where the forces are parallel the intersection-point is at infinity and the value of any one of the forces is numerically equal to the sum of the other two (algebraic sum).]

22. Example of a Three-force Piece.—The bell-crank BCD of Fig. 19 furnishes an instance. This body is subjected to the three hinge-pressures at B, C, and D, respectively. That at D is a vertical downward pressure G', equal to the weight G of the two-force piece DE. The action-line of the hinge-pressure at B has been found by the previous consideration of the two-force piece AB, and is $a\,.\,.\,b$. The hinge-pressure P'', at C, passes through the centre of the corresponding circle, but its action-line is as yet unknown. The problem, then, stands thus: Of the three forces, G', P', and P'', under whose action the bell-crank is in equilibrium, G' is known in amount and line of action, P' is known as to action-line but not in amount, while as to P'' we know neither its amount nor its direction but simply one point, C, of its action-line. However, since the three action-lines must meet in a common point, we need only note the intersection, o, of the known action-lines of P' and G', and join o with C (centre), in order to determine $C\,.\,.\,o$, the action-line of P''. Next, as to finding the amounts of both P' and P'', consider that P'' is the anti-resultant of P' and G', and that therefore the (ideal) resultant of P' and G' must act along $o\,.\,.\,C$; hence lay off $o\,.\,.\,m$ by scale to represent G' and through m draw a line $\parallel$ to $o\,.\,.\,A$ intersecting $o\,.\,.\,C$ in some point r, and draw $r\,.\,.\,k \parallel$ to G', to determine k on the line $o\,.\,.\,B$. Then $o\,.\,.\,n$, laid off along $C\,.\,.\,o$, and $= \overline{o\,.\,.\,r}$, but in the opposite direction from o, gives the amount and direction of P''. For $o\,.\,.\,r$ is the resultant of P' and G', and $o\,.\,.\,n$ is its equal and opposite.

Of course P' and P'' must be measured by the same force-scale that was used in laying off $\overline{o\,.\,.\,m} = G'$.

We can see by inspection of the figure that if the position of the link, or two-force piece, AB, were changed in such a manner that, while the line $A\,.\,.\,B$ continues to pass through o, the pin-joint, or hinge, B is caused to approach nearer and nearer to C, the forces P'' (always equal to the ideal $o\,.\,.\,r$) and P' both *increase without limit;* for the point r moves out from o, m being fixed (i.e., the load G remains invariable) until, when B is infinitesimally near to C, $m\,.\,.\,r$ is $\parallel$ to $o\,.\,.\,C$ and r is at infinity.

The method just pursued is a graphic one; *analytically*, we would proceed thus: since the system of forces G', P', and P'' is balanced, i.e., in equilibrium, the algebraic sum of their moments about any point in the plane must vanish, i.e., $= 0$.

(See § 16.) Take an origin, or centre of moments, at C and denote by d and c the lengths of the perpendiculars let fall from C upon the action-lines of G' and P', respectively. (These lengths may be obtained trigonometrically from given distances and angles, but are most easily, and with sufficient precision, scaled off from an accurate drawing.) With C as origin the lever-arm of the force P'' is zero and hence the moment of this force is zero; consequently this force does not enter the moment-equation, which therefore will contain but one unknown quantity, viz., the amount of the force P'.

The resulting moment-equation (following the routine recommended at the foot of page 33, M. of E.) is

$$P' . c - G' . d + P'' \times 0 = 0 \; . \; . \; (1); \quad \text{whence} \quad P' = \frac{d}{c} G' \; . \; . \; . \; (1)$$

becomes known; and since P'' is the anti-resultant of P' and G', we have also, α being the angle between the action-lines of the latter forces.

$$P'' = \sqrt{P'^2 + G'^2 + 2P'G' \cos \alpha} . \quad . \quad . \quad . \quad . \quad (2)$$

(See p. 7, M. of E.)

23. Other Examples of Three-force Pieces.—The ordinary *straight lever*, with flat-contact supports, is shown in Fig. 20. Since the pressures (or *reactions*) of the supports against the lever must be ┐ to the axis of the latter, and hence parallel, in this case the action-line of the third force P' must be made ┐ to the lever. *Otherwise equilibrium could not be maintained*, for the point of intersection of the three force-lines is fixed by the intersection of the *flat-contact pressures* P'' and Q; at infinity in this instance.

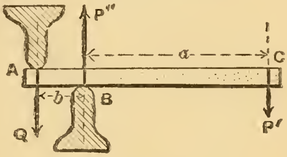


FIG. 20.

Given P', we determine Q and P'' by considering the lever as a free body under a system of three forces in equilibrium (in a plane), taking a moment-centre at B so as to exclude the unknown force P'' from the equation; and obtain first, as a moment equation,

$$Qb - P'a = 0, \; . \; . \; \text{or,} \quad Q = \frac{a}{b} P';$$

and then, by summing all the components of the three forces in a direction at right angles to the lever,

$$+P'' - Q - P' = 0; \quad \text{whence} \quad P'' = Q + P';$$

and thus the two unknown forces have been found in amount (and are already known in position).

Obviously, if Q is given, being, e.g., the weight of a body to be sustained, we compute in a similar manner the necessary force P' to be applied at C and the resulting pressure, $P'' = P' + Q$, at the support or fulcrum B.

It is also evident that the smaller the distance b is made in comparison with a, the greater the pressures P'' and Q, for a given P'; in fact, as b approaches zero, P'' and Q *increase without limit*. For a given Q and diminishing value of the ratio $b : a$, the necessary force P' decreases toward zero. (N.B. In these cases the weight of the lever itself is neglected.)

As showing how the possibility of equilibrium may be dependent in some cases on the design and position of the supporting surfaces, let us consider the curved lever in Fig. 21, where the supporting surfaces A and B are capable of furnishing only *flat-contact* pressures or reactions, whose directions, A . . O and B . . O, are fixed, being normal to the respective *smooth* and flat surfaces of contact. (N.B. Smooth surfaces are postulated in all the present problems; rough surfaces will be considered later.)

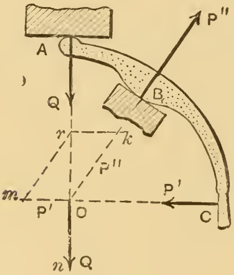


FIG. 21.

The intersection, O, of their action-lines is therefore fixed, and if a force P' is to be applied at a given point C to induce pressures at A and B, its line of action must be taken along C . . O; otherwise the lever will begin to move out of its present position (weight of lever neglected). Given, then, the force P' along C . . O, we determine P'' and Q for equilibrium, in the same manner as before shown, by filling out the parallelogram O . m . r . k in Fig. 21, precisely as was done in Fig. 19 (except that the P' of this problem corresponds to the G' of Fig. 19).

However, if we assume *any* direction at pleasure for the action-line of P' through the point C, and wish to secure equilibrium, we have only to change the mode of support at A or at B (say B), into a pin-joint or hinge-support; for the direction of a hinge-pressure is not determined solely by the nature of this mode of support; the only restriction upon it known in advance is that it must pass through the centre of the hinge-circle, its direction being determined by other relations. This change being made, we have Fig. 22, in which P' is given, or assumed, both in amount and *position*. At A we have an unknown flat-contact pressure Q in a known action-line $A\,.\,.\,O$. The intersection O of the two action-lines $A\,.\,.\,O$ and $C\,.\,.\,O$ must be a point in the action-line of P'', the unknown hinge-pressure; i.e., $B\,.\,.\,O$ is the action-line of the latter, while its amount, as well as that of Q, is found by a construction like that in Fig. 19, which need not be explained again. The results are that

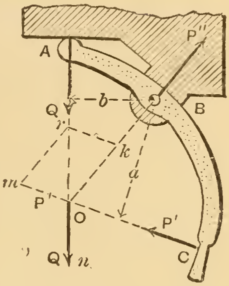


FIG. 22.

$$P'' = O\,.\,.\,k, \text{ and } Q = O\,.\,.\,n \text{ (the equal and opposite of } O\,.\,.\,r\text{)},$$

these lines, or rather lengths, representing forces on the same scale as that on which $O\,.\,.\,m$ represents the first given force P'.

24. Redundant Support.—If the two supports, A and B, of the lever are *both* hinge-joints, as shown in Fig. 23, the body or lever ABC is *redundantly supported;* for now the hinge-pressures at A and B are *indeterminate*, from simple Statics alone, but depend on the form and elasticity of the lever itself and upon the degree of looseness of fitting of the hinge-rings around the pins of the supports A and E, and upon any slight elastic yielding of the latter; as well as upon the amount and position of the force P'.

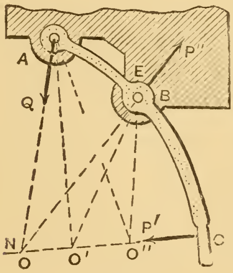


FIG. 23.

In fact, if the body is elastic, and we have to "spring" it to cause the rings to fit over the pins, pressures are produced at the supports *before the application of any force at C.*

From simple Statics, then, all that we can claim is that the action-lines of the hinge-pressures P'' and Q must intersect in some point O situated on the given action-line of the given force P'; but have no means of fixing the position of this point O. Problems of this nature, therefore, cannot be treated until the theory of Elasticity is presented, and then, as will be seen, only in comparatively simple cases. In attempting an analytical treatment in this case we should find that it presents *four* unknown quantities; whereas from simple Statics only *three* equations (independent equations) can be obtained for the equilibrium of a system of forces in a plane; hence the indetermination.

25. Four-force Pieces.—If the rigid body is in equilibrium under a system of four forces one of which is given both in amount and position, while the action-lines of all the other three are known, the amounts of those three can be determined.

A simple graphic method for solving this case is based on the obvious principle that if four forces are in equilibrium the (ideal) resultant of any two must be equal and opposite to the resultant of the other two and have the same action-line.

26. The Simple Crane.—A convenient example of a four-force piece is presented by the simple kind of crane, ABC, in Fig. 24, consisting of a single rigid body, of curved form. Its lower extremity rests in a shallow socket, while at B the edge of the (wharf) floor furnishes lateral support. We neglect the weight of the crane, and assume that no pressures are induced at A and B unless the crane bears a load; i.e., that the parts are loosely-fitting at A and B. Placing now a known load at C, viz., P_1, we note that in preventing the overturning of the crane the right-hand edge of the floor at B reacts against the crane with some horizontal pressure P_4 (*horizontal*, since the surface of contact is vertical), while at A there are two surfaces under pressure, one of which is horizontal,

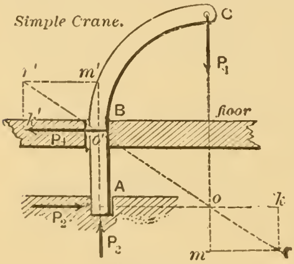


Fig. 24.

while the other is the left-hand vertical side of the socket; therefore at A we have the vertical and horizontal reactions, P_3 and P_2, both unknown in amount.

Now pair off the four forces at convenience; for example, as in our figure, pair off P_1 with P_2, noting the intersection, o, of their action-lines; P_3 and P_4 constitute the other pair and intersect at o'.

Since o is a point in the action-line of the resultant of P_1 and P_2, while o' is a point in that of the resultant of P_3 and P_4; and since these two resultants must have a common action-line for equilibrium, that common action-line must be $o\,.\,.\,o'$. Proceed therefore as follows: Prolong $C\,.\,.\,o$ and make $o\,.\,.\,m$ equal to P_1 by any convenient scale. We know that the diagonal of the parallelogram formed on P_1 and P_2 must lie on the line $o\,.\,.\,o'$ (prolonged, here). This parallelogram is found by making $m\,.\,.\,r \parallel$ to P_2 to determine r on $o\,.\,.\,o'$; then drawing $r\,.\,.\,k \parallel$ to $m\,.\,.\,o$ to intersect $A\,.\,.\,o$ in some point k. Now prolong $o\,.\,.\,o'$ beyond o', making $o'\,.\,.\,r'$ equal to $o\,.\,.\,r$. Then $o'\,.\,.\,r'$ is the resultant of the unknown P_3 and P_4, and by drawing the proper parallels, as shown, we resolve $o'\,.\,.\,r'$ in the directions of those forces and thus determine $o'\,.\,.\,m' = P_3$, and $o'\,.\,.\,k' = P_4$; $o\,.\,.\,k$, already found, is equal to P_2; all, of course, on the assumed scale of force (which scale is entirely independent of the scale for *distances* used in laying out the dimensions of the crane on the paper).

Evidently, in this particular problem, $P_1 = P_3$, and $P_2 = P_4$. (N.B. It will be noted that by this method the directions of pointing of the unknown forces are found, as well as their magnitudes.) It is also easily noted, on inspection, that if the distance $B\,.\,.\,A$ is made shorter and shorter and diminishes towards zero, P_1 remaining the same in amount and position, the forces P_2 and P_4 increase without limit (i.e., become *infinite* for $B\,.\,.\,A = 0$).

27. Simple Crane Treated Analytically. Single Rigid Body.—Take A as a centre of moments (for by so doing we exclude the two unknown quantities P_2 and P_3 from the moment-equation, since then their lever-arms are both zero; and thus obtain an equation containing only one unknown quantity). The lever-arm of P_4 is Ao', and that of P_1, Ao. Hence

$$+P_4 . \overline{Ao'} - P_1 . \overline{Ao} + 0 + 0 = 0\text{; whence } P_4 = \frac{\overline{Ao}}{\overline{Ao'}} . P_1 . . \quad (1)$$

The balancing of horizontal components gives

$$+P_2 - P_4 = 0,\quad \text{whence}\quad P_2 = P_4;\quad \text{i.e.,}\quad P_2 = \frac{\overline{Ao}}{\overline{Ao'}} . P_1 . . \quad (2)$$

Also, from the balancing of vertical components,

$$P_3 - P_1 = 0\text{; i.e., } P_3 = P_1 . . \quad . \quad . \quad . \quad . \quad (3)$$

28. Multi-force Pieces.—Since we can at most determine the amounts of only *three* unknown forces (action lines given) in a uniplanar system in equilibrium; and since to determine any forces at all there must be one force given in all of its elements (i.e., amount and position); it follows that when the system consists of more than four forces, all but three of them must be given in amount and position, and also the action-lines of the unknown three must be given, if the problem is to be a determinate one. In such a case it is a simple matter to combine the known forces into a single resultant by successive applications of the parallelogram of forces, and thus reduce the problem to that of a four-force piece, treating it then as in the last figure (Fig. 24), the resultant of all the known forces playing the part of P_1 in that figure. (N.B. To find graphically the resultant of two *parallel* forces we can use a construction like that in Fig. 10 or 11 of pp. 13 and 14, M. of E.)

29. Compound Crane on Platform-car.—As an example of the utility of the foregoing principles, let us apply them to the case of the several rigid bodies constituting the composite (or built-up) crane of Fig. 25.

Here we have an assemblage of seven rigid bodies, forming a rigid structure at rest; viz., the *jib*, ABC; the *tie-rod*, DB; the *mast*, $UADE$; the *tie-rod*, FE; the *platform*, FS; and the two *wheel-pairs*, M and R. (Each pair of wheels and its axle form together a single rigid body.) The track is level.

For simplicity we shall consider the platform alone as having weight, viz., a force G_1, applied in the centre of gravity, as shown; while the extremity C of the jib is to carry a load G. *We have given*, therefore, the two forces G_1 and G, and all angles and distances concerned, and are required to find the pressures induced at the various points of contact between the

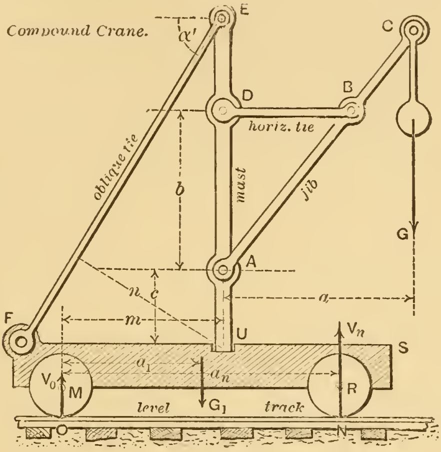


FIG. 25.

parts of the structure, and under the wheels. (The final practical object is, of course, to give sufficient strength to the parts for their respective duties, a matter, however, which cannot be entered into here, belonging, as it does, to the topic of "Mechanics of Materials.") Only analytical methods will be used.

29a. The Wheel-pressures.—Let us first consider the whole structure as a free body. The only forces *external* to this free body are the two gravity forces G and G_1, and the vertical upward pressures, V_0 and V_n, of the rails against the wheels; these constitute a system of parallel forces in equilibrium and are all shown in Fig. 25. (NOTE.—The pressures between any two contiguous parts of our present free body do not need to appear in this system for the reason that, if introduced, they would form a balanced system among themselves and might therefore be removed without affecting equilibrium. *For example*, at D the ring of the tie-rod presses horizontally and to the right against the pin of the mast; and the pin, from the principle of action and reaction, presses the ring with an equal horizontal pressure toward the left; but *both the ring and the pin* belong to the free body now under consideration in Fig. 25, and if one of these pressures is inserted in the system the other must be placed there with equal right, and *the two then annul each other's influence in all the equations of equilibrium.* We therefore conclude, in general, that *the mutual actions between the parts of a given free body in equilibrium may be omitted in applying the conditions of equilibrium.*)

With O as a centre of moments, then, we have for the free body of Fig. 25 (see figure for notation)

$$V_n a_n - G(a+m) - G_1 a_1 = 0; \quad \therefore\ V_n = \frac{G(a+m) + G_1 a_1}{a_n};$$

while by summing vertical components

$$V_0 + V_n - G - G_1 = 0; \quad \text{hence} \quad V_0 = G + G_1 - V_n.$$

29b. Pressures at the Joints; D, B, and A (Fig. 25).—By inspection we see that the tie-rod DB is a two-force piece (its own weight neglected); that is, the hinge-pressures at D and B must have a common action-line, viz., $D\ .\ .\ B$. DB is a *straight two-force piece;* or "*straight link*," as we shall hereafter call it; and is subjected to a tensile action along its axis (tension, here; as we note by inspection; a straight link under compressive action is called a *strut*, or *compression-member*).

The Jib as a Free Body. Fig. 26. It is a three-force piece, being acted on by the known vertical downward force G at C, by an unknown horizontal force T (directed toward the left) at B (T being the pull of the tie-rod), and an unknown hinge-pressure P at A, making an unknown angle α with the horiozntal.

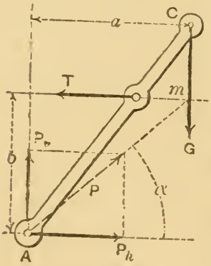


FIG. 26.

NOTE.—Since the mast which acts on the jib at A is *not a two-force piece*, we have no means of knowing the position of the action-line of P from a mere inspection of the mast, as we did with the tie DB and the hinge-pressure at B. Of course, graphically, P passes through the point m where the action-lines of the other two forces intersect; but as we are now using analytical methods, we shall replace P, which is unknown in amount and in position, by its (ideal) horizontal and vertical components, P_h and P_v (i.e., by two unknown forces in *known* action-lines). We thus have a four-force piece to deal with.

If we take A as a centre of moments, the force T will be the only unknown quantity in the corresponding moment-equation, which is $Tb - Ga = 0$; whence we have, for the tension T in the tie-rod, $T = G(a \div b)$. (This T is the value of the hinge-pressure at D and also that at B, in Fig. 25.)

Assuming an axis X horizontal and Y vertical, we now have from $\Sigma X = 0,\quad P_h - T = 0$; or, $P_h = T$; while from $\Sigma Y = 0,\quad P_v - G = 0$; or, $P_v = G$; and we now easily find P itself, since $P = \sqrt{P_h^2 + P_v^2}$; while $\tan \alpha = \dfrac{P_v}{P_h}$.

29c. Tension in the Tie-rod FE. As with DB, so with FE we note by inspection that it is a two-force piece, so that the hinge-pressure, T', at E, Fig. 25, though unknown in amount as yet, must have $E\,.\,.\,F$ as action-line. The force T' and the pressures (or supporting forces) H and V exerted by the left-hand side and bottom, respectively, of the *shallow* socket U against the foot of the mast, are the three unknown forces in known action-lines acting on the free body shown in Fig. 27, consisting of the jib, mast, and tie-rod DB. The mutual actions between these three bodies are *internal* to the free body taken and are hence omitted (see note in § 29*a*), the external system consisting of T', H, V, and G; all in known action-lines, G being the only force known in amount. By moments about the point U, we have

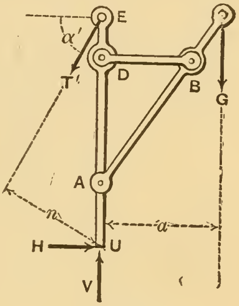


FIG. 27.

$$T'n - Ga = 0; \quad \text{whence } T' = \frac{a}{n} \cdot G. \quad . \quad . \quad . \quad (1)$$

From $\Sigma X = 0$, $H - T' \cos \alpha' = 0$, and $\therefore H = \dfrac{a}{n} G \cos \alpha'$. (2)

From $\Sigma Y = 0$. $V - G - T' \sin \alpha' = 0$; $\therefore V = G + T \sin \alpha'$. (3)

It will be noticed that we have not made use of the mast as a separate free body. This might have been done as a means of finding the three forces just determined, T', H, and V; since the hinge-pressures at D and A had already been deduced; but the process would have been more roundabout.

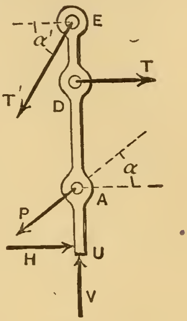


Fig. 28.

However, as a reminder of the principle of action and reaction and of the definition of force, Fig. 28 is presented, showing the system of forces we should have to deal with in treating the mast as a free body; and also Fig. 29, representing the car-platform and the two pairs of wheels as a single free body, with the external forces acting. The student will note that the H and V in Fig. 28 are the equals and opposites, respectively, of those in Fig. 29. A similar statement may be made for the T' of those figures. Again, the P and T of Fig. 28 are the equals and opposites of the P and T of Fig. 26 (action and reaction). The plane of the crane being supposed to be midway between the two wheels of each pair, the pressure V_0 is equally divided between the two wheels forming the pair on the left. Similarly at N, with V_n.

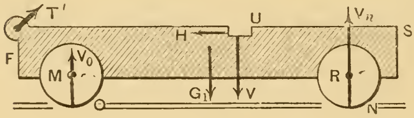


Fig. 29.

30. Simple Roof and Bridge Trusses; Ritter's Method of Sections.—A truss is an assemblage of straight pieces jointed together in one plane. If the joints consist of pins (one in each joint) inserted through holes or rings in the ends of the pieces (or "members") the truss is said to be "*pin-connected*"; while if the ends of the pieces meeting at each joint are rigidly riveted together (a favorite method in Europe) the truss is said to have *riveted connection.*

In the first case, *pin-connection*, each piece is free to turn about the pin, independently of all other pieces, during the gradual, though slight, change of form which the truss undergoes in the gradual settling of a load upon it, and the stresses induced in the pieces are called "primary stresses" (whereas, with riveted joints, other, and additional, stresses, called "secondary stresses," are caused in the pieces, from the constraint exerted on each other by the members meeting at each joint).

Confining ourselves to the consideration of *pin-connected* trusses, constructed so that *each piece connects no more than two joints*, and *loaded only* at the *joints* (its own weight being considered as concentrated at the various joints), we note that in such a case each member must be a *straight two-force piece*, or *straight link* (neglecting its own weight); i.e., it is subjected to a simple tension or compression (according as it is acting as a tie or a strut) along its axis.

NOTE.—If such a straight link be conceived divided into two parts (for separate treatment) by any imaginary transverse plane or surface passing between the joints connected by that piece, the action or force exerted on one of these parts by the other is a pull (if tension) or thrust (if compression), applied at the section and directed *along the axis or central line* of the piece. *For example*, in the truss of Fig. 30, pin-connected, and composed of "straight links," if we wish to consider free the portion on the left of the imaginary cutting surface $A\,.\,.\,B$, the system of forces acting on the ideal body so obtained (see Fig. 31) consists of the abutment reaction V_0, the loads G, G', and G'' (at the joints a, b, and c), and the three forces (pulls or thrusts) P, Q, and T, acting at the (cut) ends of the three pieces intersected by this surface $A\,.\,.\,B$.

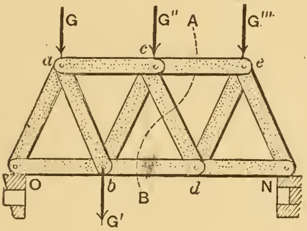


FIG. 30.

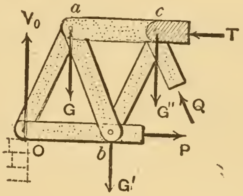


FIG. 31.

If the pier reactions V_0 and V_n have been already determined by a consideration of the whole truss as a free body, V_0 is now a known quantity, and we may go on to find the values of the three stresses (i.e., pulls or thrusts) P, Q, and T, by methods already illustrated. It is evident from inspection that the force or stress in the lower horizontal piece is a pull (P); and that the stress (T) in the upper horizontal piece is a thrust; but as to which way the arrow indicating the stress Q in the oblique member (" web-member") should be pointed (a matter not always to be decided on mere inspection) the detail of the analysis would show, provided numerical data were given throughout. Examples of the application of Ritter's method will be given later.

This method is peculiarly well adapted to the treatment of pin-connected, straight-linked, trusses, since the stress in a straight link (its own weight being neglected) is always a simple tension or compression; in other words, a pull or thrust directed along the axis of the piece.

The three stresses, P, Q, and T, which were stated to be obtainable from the free body in Fig. 31, in the three straight links concerned, could also be determined from the consideration, as a free body, of *the other portion of the truss*, viz., that on the right of the cutting surface $A \,..\, B$ of Fig. 30; in fact, in the present case with *greater simplicity*, since in this new free body, shown in Fig. 32, the system of forces in equilibrium is much simpler, though there is, of course, the same number of unknown quantities, P, Q, and T; which, it is to be carefully noted, are the *equals and opposites, respectively, of the* P, Q, and T of Fig. 31.

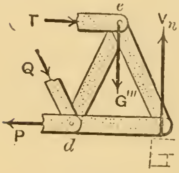


FIG. 32.

31. **Remark.**—From the foregoing illustrations (dealing with compound quiescent structures bearing loads) we note that in the various free bodies (whose conception has been necessary for introducing the different unknown forces into balanced systems) the pressures or pulls of any two parts of the *same* free body against *each other are omitted* from the system; and also (see last paragraph) that a portion of a two-force piece, situated at one end thereof, may be conceived removed, and the pull or thrust of that portion against the remaining portion inserted in the system, *provided it is a straight two-force piece, whenever it is desired to consider separately either part of a pin-connected truss,*

conceived to be divided into two parts by a cutting plane or surface (see the last three figures). (The stresses in bodies that are not straight links will be considered later in the "*Mechanics of Materials*," pp. 195–514, M. of E.)

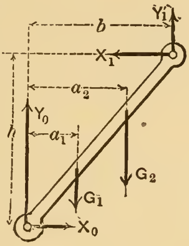


FIG. 33.

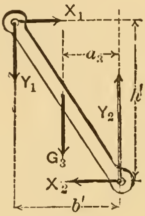


FIG. 34.

32. Problem of the Two Links (Prob. 2 and Fig. 36 of p. 35, M. of E.).—Consider link AB free in Fig. 33. Since this is not a two-force piece, the action-lines of the hinge-pressures at the extremities do not coincide, nor does either follow the axis of the piece. Hence these pressures are best represented by their horizontal and vertical components, as shown, so that the four unknown quantities, X_0, Y_0, X_1, and Y_1, are to be determined. Similarly, considering the other link as free, we have Fig. 34, in which it is to be noted that the components of the pressure at the upper hinge *are respectively equal and opposite* to those (X_1 and Y_1) of Fig. 33 (action and reaction), so that there are only six unknown "force-amounts" in the two figures, instead of eight, as might appear at first sight. Hence the problem is determinate, since from each of these free bodies we obtain three independent equations; or six in all, as follows: Putting $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma(Pa) = 0$ (i.e., Σ moments), for each body in turn, taking the centre of moments at the lower hinge in each case, we have

$$\text{For Fig. 33} \begin{cases} X_0 - X_1 = 0; \; Y_1 + Y_0 - G_1 - G_2 = 0; \\ \text{and } X_1 h + Y_1 b - G_1 a_1 - G_2 a_2 = 0. \end{cases}$$

$$\text{For Fig. 34} \begin{cases} X_1 - X_2 = 0; \; Y_2 - Y_1 - G_3 = 0; \\ \text{and } Y_1 b' - X_1 h' + G_3 a_3 = 0. \end{cases}$$

The elimination, by which to find separately the six unknowns, X_0, Y_0, X_1, Y_1, X_2, and Y_2, is left to the student. (Treated graphically, this case would come under Class B of p. 458, M. of E.)

33. Problem of Rod and Cord (Prob. 4 and Fig. 41 of p. 37 of M. of E.).—Consider the rod free by cutting the cord and removing the pin of the hinge; that is, besides the forces G_1 and G_2, we must insert the unknown tension P along the axis of the cord, at an angle α with the rod, and the horizontal and vertical components, X_0 and Y_0, of the hinge-pressure, whose action-line is unknown, and thus have a complete system of forces in equilibrium. There are only three unknown quantities, P', X_0, and Y_0.

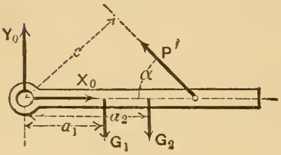


FIG. 35.

From Σ hor. comps. $=0$, we have $X_0 - P' \cos\alpha = 0$; . . . (1)
" Σ vert. " $=0$, " " $Y_0 + P' \sin\alpha - G_1 - G_2 = 0$; (2)
Σ (moms. about hinge) . . . $P'c - G_1a_1 - G_2a_2 = 0$. . . . (3)

Since eq. (3) contains only one unknown, P', we have at once

$$P' = (G_1a_1 + G_2a_2) \div c;$$

and knowing P', we obtain X_0 from (1), and Y_0 from (2); and finally the hinge-pressure, $R = \sqrt{X_0^2 + Y_0^2}$, making an angle whose tan $= Y_0 \div X_0$ with the horizontal.

34. Problem of Simple Roof-truss (Prob. 5 and Fig. 40 of p. 37, M. of E.).—The right-hand support is supposed to furnish all the horizontal resistance. Hence the system of forces acting on the whole truss, considered free, will be as shown in Fig. 36, in which there are three unknown reactions (or pressures, of the supporting surfaces), V_0, V_n, and H. H becomes known from

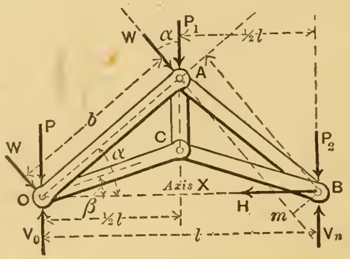


FIG. 36.

$$\Sigma X = 0, \quad \text{viz.,} \quad 2W \sin\alpha - H = 0. \quad . \quad . \quad . \quad . \quad (1)$$

By moments about point O, we have

$$V_nl - P_2l - P_1 \cdot \tfrac{1}{2}l - Wb + 0 + 0 + 0 = 0, \quad . \quad (2)$$

which can be solved for V_n, while from Σ moms. about B

$$-V_0l + Pl + W \cdot l\cos\alpha + W(l\cos\alpha - b) + P_1 \cdot \tfrac{1}{2}l = 0, \quad (3)$$

from which V_0 can be obtained.

[NOTE.—Having now made use of three independent equations, based on the laws of equilibrium of forces, and by their aid determined three quantities originally unknown, the student should not imagine that by putting $\Sigma Y = 0$, or by writing another moment summation about the point A (for example), he thereby secures another independent equation, from this same free body, capable of determining a fourth unknown quantity. He would find that such an equation could be established by mere algebra, from the first three above, without further reference to the figure, and hence would be useless as regards determining any other unknown quantities. See top of p. 33, M. of E.]

All loads or forces being considered to act at the joints, and no piece extending beyond a joint, we note that this roof-truss is composed entirely of *straight two-force pieces* (each in simple tension or compression along its axis), so that portions of the truss may be considered free, isolated by the passing of one or more cutting surfaces. For example, to find the stress in piece AO and that in CO consider the free body in Fig. 37, where S and T are the stresses required. The forces in this figure form a concurrent system, for which

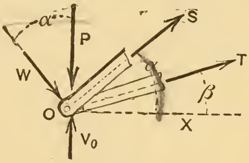


FIG. 37

$$\Sigma X = 0 \text{ gives } S\cos\alpha + T\cos\beta + W\sin\alpha = 0, \quad . \quad . \quad . \quad (4)$$

$$\text{and } \Sigma Y = 0 \quad \text{“} \quad S\sin\alpha + T\sin\beta + V_0 - P - W\cos\alpha = 0. \; . \, (5)$$

Solve these for S and T. In a numerical case one or both of these will come out negative, indicating compression, not tension as assumed in the figure.

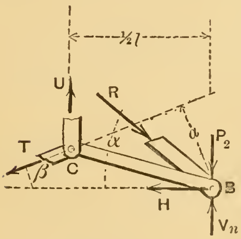


FIG. 38.

To find the stresses in AC and AB, the free body shown in Fig. 38 may be taken. Here the forces form a non-concurrent system. Taking moments about B, we have

$$Ta - U \, . \, \tfrac{1}{2}l + 0 + 0 = 0; \quad . \quad (6)$$

from which, T being already known, U can be obtained. For R, put $\Sigma Y = 0$, and it will be the only unknown quantity; or, put moments about $C = 0$.

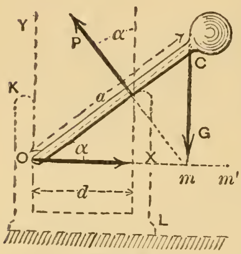


FIG. 39.

35. Problem. Rod and Tumbler (Fig. 39).—The tumbler is smooth-edged with vertical sides. The rod has smooth sides and weighs G lbs. C is its centre of gravity, at a distance a from the end of rod. Given the distances a and d, at what inclination α with the horizontal should the rod be placed, in contact with tumbler at two points as shown, that the position of the rod may be *stable*, i.e., that the rod may remain in equilibrium? G, a, and d are known; H, P, and α unknown; H and P being the pressures of the tumbler against the rod at the two points of contact. H must be horizontal (why?), and P ┐ to side of rod and hence at angle α with the vertical.

The rod being the free body,

from $\Sigma X = 0$ we have $H - P \sin \alpha = 0$; . . . . (1)

" $\Sigma Y = 0$ " " $-G + P \cos \alpha = 0$; . . . . . (2)

" Σ(moms. about O) we have $Pd \sec \alpha - Ga \cos \alpha = 0$. . (3)

Now $\sec \alpha = 1 \div \cos \alpha$, and, from (2), $G = P \cos \alpha$. Hence (3) becomes

$$P\frac{d}{\cos \alpha} - Pa \cos^2 \alpha = 0; \quad \therefore \cos \alpha = \sqrt[3]{\frac{d}{a}}.$$

α being now known, P and H are easily found from (2) and (1). (A brief mode of finding α alone is based on the fact that the three action-lines concerned must meet in a common point m. If, therefore, a figure be drawn in which the action-line of P intersects that of H in a point m' not coincident with m, that of H and G, we have only to form a trigonometrical expression for the distance $\overline{mm'}$, involving a, d, and α, write it $= 0$, and solve for $\cos \alpha$ or $\sec \alpha$.)

36. Problem. Pole and Tie (Fig. 40).—Given the load P, the weight G of the pole, and all the distances and angles marked in the figure, it is required to find the tension P' induced in the chain, whose weight is neglected, and which thus serves as a straight tie. The pole is hinged at O.

The pole may be considered free, as already shown in the

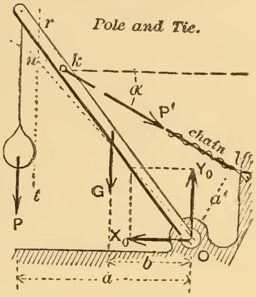


Fig. 40.

figure, by inserting the pull P' exerted on it at k in a known action-line, $k\,.\,.\,l$, by the chain, and the horizontal and vertical components, X_0 and Y_0, of the hinge-pressure at O. The action-line of this hinge-pressure makes an unknown angle with the horizontal, but must pass through the centre of the hinge at O. There are three unknowns in the system, viz., X_0, Y_0, and P'.

From $\Sigma X = 0$ we have

$$P' \cos \alpha - X_0 = 0 ; \quad . \quad . \quad (1)$$

$$\text{“} \quad \Sigma Y = 0 \quad \text{“} \quad \text{“} \quad Y_0 - P - G - P' \sin \alpha = 0 ; \quad . \quad . \quad (2)$$

$$\text{“} \quad \Sigma (\text{moms. about } O) = 0,\ Pa + Gb - P'a' = 0. \quad . \quad . \quad (3)$$

The value of P' is easily found from (3), then that of X_0 from (1), and that of Y_0 from (2). Hence the amount of the hinge-pressure, $P_0 = \sqrt{X_0^2 + Y_0^2}$, becomes known, and the tangent, $= Y_0 \div X_0$, of the angle between its action-line (On) and the horizontal.

Graphic Solution. The action-line of the resultant, R, $= P + G$, of the known parallel forces P and G may easily be determined by a construction like that of Fig. 10, M. of E.; or by applying the principle of the foot-note, p. 14, M. of E. Since this action-line intersects that of the force P' in some point n, the hinge-pressure at O must act along the line $O\,.\,.\,n$ and must be the equal and opposite of the resultant of P' and R.

37. Problem. Three Cylinders in Box (Fig. 41).—Three solid homogeneous circular cylinders, of equal weight, $= G$, and of the same dimensions, rest in a box, as shown, of horizontal bottom and vertical sides. The two lower cylinders barely touch each other; i.e., there is no pressure between them, as the box is an easy fit. The centres of the three cylinders form the vertices of an equilateral triangle. It is required to find the pressures at all points of contact (*points*, in

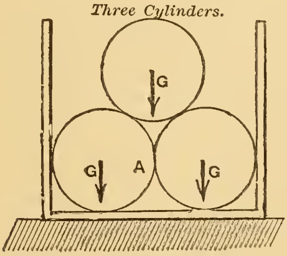


Fig. 41.

this *end view ;* really, *lines* of contact) between the cylinders and the box; also between the cylinders themselves. All surfaces smooth.

Fig. 42 shows the upper cylinder as a free body, there being three forces acting on it, viz., G, the action of the earth, or gravity, and the two pressures, P' and P'', from the cylinders beneath. From symmetry, P' and P'' must be equal. From Σ (vert. comps.) $= 0$,

$$2P' \cos 30° - G = 0\,;$$

$$\text{i.e.,}\quad P' \text{ (or } P'') = G \div \sqrt{3}.$$

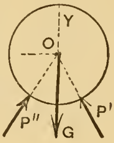


FIG. 42.

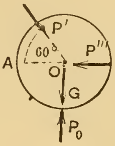


FIG. 43.

Taking now the lower right-hand cylinder free in Fig. 43, we find it under the action of its weight G; of the pressure P' (the equal and opposite of the P' in Fig. 42) now known; of the unknown horizontal pressure or reaction P''' from the side of the box; and of the vertical pressure P_0 from the bottom, also unknown. (Concurrent system, with two unknown quantities.) There is no pressure at A. (See above.) From Σ (hor. comps.) $= 0$,

$$P' \sin 30° - P''' = 0\,; \quad \therefore P''' = G(\sqrt{3} \div 6).$$

From Σ (vert. comps.) $= 0, P_0 - G - P' \cos 30° = 0\,; \therefore P_0 = \frac{3}{2}G.$

It thus appears that the sum of the two pressures on the bottom of the box is $3G$, i.e., is equal to the combined weight of the three cylinders; if the sides of the box were not vertical, however, this would not be true, necessarily.

38. Problem of the Door.—Fig. 44 shows an ordinary door as a free body. Support is provided by two vertical hinge-pins, of smooth surfaces, whose distance apart is such that the *lower one alone* receives vertical pressure from the door (i.e., furnishes a vertical supporting force, V_0, at its horizontal upper face). The upper hinge-pin provides only lateral support, as seen in the horizontal reaction H which prevents the door from falling away to the right, from that hinge. Similarly, the right-hand vertical edge of the lower

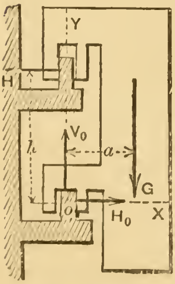


FIG. 44.

hinge-pin, by its reaction H_0, offers lateral support at that point. Given the weight G of the door (considered as concentrated in its centre of gravity) and the distances a and h, it is required to determine the three pressures, H, H_0, and V_0.

From Σ(vert. comps.) $= 0$ we have $V_0 - G = 0$; i e., $V_0 = G$.

From Σ (moms. about O), $Hh - Ga = 0$; i.e., $H = \frac{a}{h}G$.

From Σ (hor. comps.) $= 0$, $H - H_0 = 0$; i.e., $H_0 = \frac{a}{h}G$.

Since V_0 is parallel and numerically equal to G, and H to H_0, the system of forces acting on the door is seen to consist of two couples of equal moments of opposite sign, thus balancing each other. The smaller the distance h the greater the value of H (and of its equal, H_0), if G and a remain unchanged.

(Unless the fitting of the parts is very accurate, only one hinge of a door receives vertical pressure—i.e.. carries the weight, in practical language.) If more than one receives vertical pressure, the share carried by each depends on the accuracy of fitting and on the slight straining or change of form of the parts under the forces acting.)

39. Problem of the Wedge and Block (Fig. 45).—The shaded parts represent a smooth horizontal table or bed-plate eh, and a flat and smooth vertical guide md, both immovable. W is the wedge whose angle of sharpness is α, and B the block, on which rests a weight (not shown) whose pressure on B is vertical and $= Q$. The weights of wedge and block are neglected. There is supposed to be no friction, otherwise the results would be quite different (see § 9 of Graph. Stat. of Mechanism, in this book, and p. 171, M. of E.). Given Q and α, what force P must be applied horizontally at the head of the wedge to prevent the block B from sinking (or to raise the block with *constant velocity* if the latter has an upward motion)?

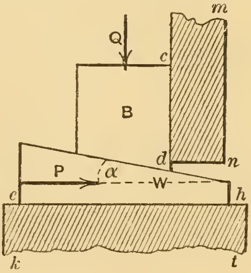


FIG. 45.

Supposing the required force P to be in action, and that there is no friction, the mutual pressure, N, between block and wedge is normal to the surface of contact and hence makes an angle α with the vertical, while the pressures on surfaces cd and eh, viz., S and R, are horizontal and vertical, respectively. Hence, when the block B is considered free (Fig. 46) we have equilibrium between the three forces, Q, N, and S; the two last are unknown, but are determined thus:

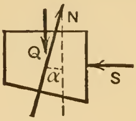


FIG. 46.

$$\text{From } \Sigma\,(\text{hor. comps.}) = 0, \quad S = N \sin \alpha;$$

$$\text{From } \Sigma\,(\text{vert. comps.}) = 0, \quad N \cos \alpha = Q.$$

Similarly, we have the wedge free in Fig. 47 under the action of N, now known, and the unknown R and the required P. Hence

$$R = N \cos \alpha, \ (= Q), \text{ from } \Sigma Y = 0;$$

$$\text{and} \quad P = N \sin \alpha, \ = Q \tan \alpha, \text{ from } \Sigma X = 0.$$

Evidently, the sharper the wedge the smaller the force P necessary for a given Q.

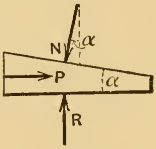


FIG. 47.

40. Cantilever Frame (Fig. 48).—This frame consists of eleven straight pieces in a vertical plane, pin-connected, and supposed without weight. Pieces C, D, E, and I are horizontal, G and H vertical, the others oblique. The vertical rod at A is anchored, as shown, but no stress is induced in it (nor in any other piece) unless a load P is placed at w (no other joint to be loaded). Since, furthermore, no piece connects more than two joints, *each piece is a straight two-force piece*, subjected to a tensile or compressive stress along its axis, and any one may be conceived to have a portion removed, when desired, in the isolation of the various "free bodies" to be considered. The necessity of the rod and anchorage is evident from the fact that the rectangle $m'nom$ is left unbraced. The load P being given, it is required to find the stress in each piece of the frame and in the rod at A.

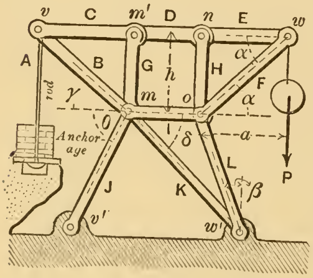


FIG. 48.

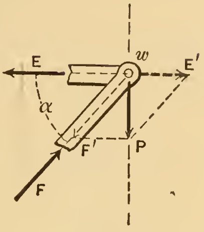


FIG. 49.

The free body in Fig. 49 enables us to find the stresses E and F. Since only three forces are concerned, meeting at a point, a simple procedure is to resolve the known P into two components, E' and F', along the action-lines of E and F, as shown by dotted lines. E' and F' are equal and opposite to the required stresses E and F, respectively; i.e., $E = P \cot \alpha = P\frac{a}{h}$, and $F = P \operatorname{cosec} \alpha = P \div \sin \alpha$. (The summation of horizontal and vertical components put equal to zero would give the same result.) E is tension; F, compression.

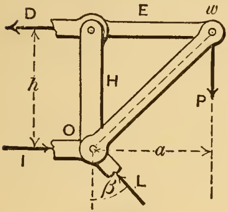


FIG 50.

Next take the free body in Fig. 50, involving the known force P and the unknown stresses D, I, and L, assuming for them the characters implied by the pointing of the arrows. Taking moments about centre of pin at O, we have

$$Dh - Pa = 0; \quad \text{whence } D = +P\frac{a}{h},$$

and is tension.

From Σ (vert. comps.), $P = L \cos \beta$, or $L = +\frac{P}{\cos \beta}$, and is therefore compression as assumed; while, from Σ (hor. comps.) $= 0$, $I - D - L \sin \beta = 0$; i.e., $I = +P(\tan \beta + \cot \alpha)$, and is compression.

To find the stresses in A, B, and G we use the free body in Fig. 51. By moments about O,

$$Ad - Dh = 0;$$

$$\therefore A = +P\frac{a}{d}, \text{ and is tension.}$$

From Σ (hor. comps.) $= 0$,

$$D - B \cos \gamma = 0;$$

whence $B = +\frac{a}{h} \cdot \frac{P}{\cos \gamma}$, and is compression, as assumed.

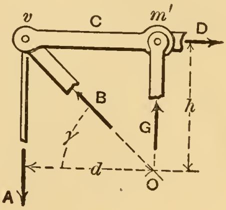


FIG. 51.

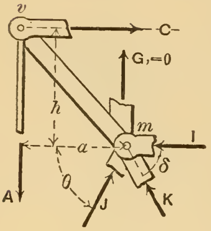


FIG. 52.

In Fig. 48 we note that since pieces C and D are in the same straight line and G is the only other piece connecting with the joint m' (there being also *no load* on m'), the stress in the piece G must be zero and the stress in C must equal that in D; similarly, the stress in piece H is zero and $E = D$ (as already found above). Hence G is marked $= 0$ in Fig. 52, showing a free body in which the stresses J and K are the only unknown forces, C being $= D, = P\frac{a}{h}$, and $G =$ zero. Therefore, by elimination between the equations

$J \cos \theta + C - I - K \cos \delta = 0$ (from sum of hor. comps.), and
$J \sin \theta + K \sin \delta - A = 0$ (from sum of vert. comps.),

we obtain the stresses J and K, both of which are assumed to be compressions in this figure.

If the joints m' and n were not both in the horizontal line joining w and v, Fig. 48, the stresses in G and H would not be zero, as they are in this instance.

40a. Real and Ideal Forces. "Balanced Forces."—The student should be careful to distinguish between *real* and *ideal* forces. If a body A receives pressures P and Q from bodies B and C respectively, the resultant of P and Q is purely *ideal*, being merely *conceived* to take the place of P and Q, if any useful and legitimate purpose can thereby be served. Again, the X and Y components of an actual, or *real*, pressure P are *ideal*, serving a mathematical purpose only, when we suppose P to be removed and its components inserted *in its stead*.

Such customary phrases as "balanced forces," "forces in equilibrium," etc., are unfortunately worded, as they seem to imply that forces act on forces, which is an absurdity. In reality, *bodies act on bodies*, *force* being the mere name given to the *action* (if it is a push, pull, etc.); so that, instead of stating that "the forces are balanced," we should more logically say: "The rigid body is in equilibrium, or balanced, under the actions of certain other bodies." (See correspondence in the London *Engineer* of June, July, and Aug., 1891.)

CHAPTER III.

MOTION OF A MATERIAL POINT.

41. Velocity and Acceleration.—Any confusion between the ideas of velocity and acceleration is fatal to a clear understanding of the subject of motion. Just as velocity may be defined as *the rate at which distance is gained*, so acceleration may be defined as *the rate at which velocity is gained*, thus: If at a certain instant a material point has a velocity of 10 feet per second, we mean that it has such a speed of motion that it *would* pass over a distance of 10 feet *if* that rate of speed remained constant during the whole of the next second of time. Similarly if, at the same instant, the acceleration of the material point is said to be 20 feet per "square second" (or "20 feet per second, per second"), we mean that *if* the velocity were to continue to change, during the whole of the next full second, at the same rate at which it is changing at the instant mentioned, the velocity at the end of that second *would* be greater by 20 units of velocity than at the instant mentioned, i.e., the velocity would be 30 feet per second. According to this definition, then, if the velocity of the material point remains unchanged, the acceleration is zero at every point of the motion.

At each point, then, in the path of a material point moving along a right line, we have to do with

s, its distance from some convenient origin in that line;

v, its velocity, or rate at which that distance is changing, $= \dfrac{ds}{dt}$;

p, its acceleration, or rate at which the velocity is changing, $= \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$.

There is no need of defining still another quantity as the rate at which the acceleration is changing, for the reason that the force which at any instant occasions the peculiarity of the motion of the material point is determined from the acceleration, viz., from the relation *force* = *mass* × *accel.*

In the uniformly accelerated motion of a free fall *in vacuo*, the following are the values of the above quantities at beginning of the motion, and at the end of each full second thereafter, thus:

	s, = Distance.	v, = Veloc.	p. = Acceleration = g.
At beginning, . . .	0.0 ft.	0.0 ft. p. sec.	32.2 ft. p. (sec.)^2;
At end of 1st full sec.,	16.1 ft.	32.2 “ “	32.2 “ “
At “ 2d “ “	64.4 ft.	64.4 “ “	32.2 “ “
At “ 3d “ “	144.9 ft.	96.6 “ “	32.2 “ “

A recent English writer is so desirous that acceleration shall not be confused with velocity that he calls the unit of velocity a “*Speed*” and the unit of acceleration a “*Hurry*.” For example, using the foot and second as units, he would say that at the end of the second full second of a free fall *in vacuo* the velocity is 64.4 “*speeds*,” while the acceleration at the same instant is 32.2 “*hurries*.” These words are quite suggestive and should be borne in mind.

To say that at a certain instant the acceleration is zero does not imply that the body is not moving, but simply that its velocity, however small or great, is not changing; and again, the statement that the velocity is zero at a certain instant does not imply that the acceleration is zero also, but only that the velocity, as its value changes, is then passing through the value zero, while the rate at which it is changing (i.e., the acceleration) is not determined until some further statement is made.

42. Momentum.—This word is used quite largely in some works on Mechanics, but may be considered a superfluity, liable to give rise to confusion of ideas; though sometimes useful, it need rarely be used. By definition, it is a name given to the product of the *mass of a material point by its velocity at any instant*, i.e., Mv. Of course, the mass of the body is a constant quantity, while its velocity may be continually changing; hence the momentum is always proportional to the velocity. In accordance with this definition, the value of a force which is accelerating the velocity of a material point in its path is sometimes stated to be equal to the *rate at which the momentum is changing;* by which is simply meant the following:

From eq. (IV), p. 53, M. of E., we have $P = Mp$, $=$ (*mass* × *accel.*); but $p = \frac{dv}{dt}$; hence we may write

$$P = M\frac{dv}{dt} = \frac{Mdv}{dt} = \frac{d[Mv]}{dt} = \begin{bmatrix}\text{change of momen-}\\ \text{tum in time } dt\end{bmatrix} \div dt;$$

i.e., $P =$ *the rate at which the momentum is changing.* Therefore the above statement as to the value of the accelerating force is nothing more than what we already have in the form of $P = Mp$.

43. Cord and Weights.—There are few mistakes more common than rushing to the conclusion that the tension in a vertical cord to which a weight is attached is equal to that weight. This may be true (and *is true if the weight is at rest* and *has no other* support), but should not be assumed without thought.

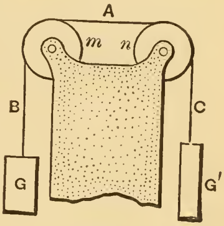


FIG. 53.

For example, Fig. 53, having two weights attached to the same cord, if the point A of the cord were fastened, the tension in B would be $= G$; and that in C, $= G'$, if G and G' are the respective weights of the two bodies. But if, the cord being continuous and not fastened, with no friction at the pulley-axles, an accelerated motion begins (assume $G' > G$), the tension on each side depends on *that acceleration, as well as on the weights of the bodies.* To find this acceleration, which is common to both sides, neglecting the masses and weights of the pulleys and cord (by which we mean that the tension S in the cord at A may be taken equal to that at B and C) let us consider the body G' free. We note that this body has a downward accelerated motion, and that the forces acting on it are G', directed vertically downward, and S, the tension in the cord, pointing vertically upward, i.e., acting as a resistance; hence, calling the acceleration p, we have $G' - S = \frac{G'}{g}p$. As for the other weight, it is rising with an upward acceleration $= p$, under an upward force S and a downward resistance G, whence $S - G = \frac{G}{g}p$. From these two relations we obtain by elimination

$$p = \frac{G' - G}{G' + G} \cdot g \quad \text{and} \quad S = \frac{2GG'}{G' + G}.$$

The value of p might have been obtained directly by considering that the acceleration of the motion is just the same as if the whole mass $\frac{G + G'}{g}$ were moving in the same right line under the action of a single accelerating force $G' - G$.

44. Example. Lifting a Weight.—A rigid mass weighing 100 lbs. is to be lifted vertically through a distance of 80 ft. in 4 seconds of time, and with uniformly increasing velocity, from a condition of rest (i.e., veloc. = 0). What tension must be maintained in a vertical cord attached to it, to bring about this result?

The average velocity is 20 ft. per second, and since the initial velocity is zero and the *rate of increase* of velocity (i.e., the *acceleration*, p) is to be constant (uniformly accelerated motion), the final velocity will be double the average, viz., 40 ft. per sec. (see also eq. (4), p. 54, M. of E.). If, then, 40 velocity-units are gained in 4 seconds, the acceleration is $40 \div 4 = 10$ ft. per second per second (10 "hurries"), and an upward accelerating force of $Mp = \frac{100}{32.2} \times 10 = 31$ lbs. must be provided. But the only forces actually present, acting on this mass, are the action of the earth, viz., 100 lbs. pointing vertically downward, and the tension, S, in the cord, directed vertically upward; and their (ideal) resultant, which is $S - 100$ lbs. (since they have a common action-line), is required to be 31 lbs. and to act upward; whence we have the required cord-tension, $= S$, $= 131$ lbs.

To lift the 100-lb. weight, then, under the conditions imposed, requires a cord-tension of 131 lbs. [If the cord be allowed to become slack at the end of the 80-ft. distance, the body does not immediately come to rest, as it then has an upward velocity of 40 ft. per second. Its further progress is an "upward throw"

(p. 52, M. of E.) with 40 ft. per second as an initial velocity, the only force acting at this stage being the downward attraction of 100 lbs., which will gradually reduce the velocity to zero.]

If a smaller height of lift had been assigned with the given time, or the same height with a longer time, the difference between the requisite tension S and the gravity force (or weight) of 100 lbs. would have been smaller; and *vice versa*.

It is convenient to note that in using the foot-pound-second system of units for force, space, and time, the mass of a body is obtained by multiplying its weight in pounds by 0.0310 (which is the reciprocal of 32.2); thus, in the preceding example the mass of the 100-lb. weight is $= 100 \times .0310 = 3.10$ mass-units.

45. Harmonic Motion.—From eq. (3), p. 59, M. of E., remembering that c and a are constants, we note that s, the "displacement," or distance of the body from the origin (middle of the oscillation), is proportional to the sine of an arc or angle, and that *this arc is proportional to the time*, t, elapsed since leaving the origin; whence arise the following graphical relations: In Fig. 54, let O be the origin or middle of the oscillation, and the horizontal line CD the path of the body; OD, $= r$, being the extreme displacement, i.e., the "semi-amplitude," of the motion. At any instant of time the body is at some point m between C and D, i.e., $Om =$ the variable s (displacement). With O as centre, and OD, $= r$, as radius, describe a circle and erect a vertical ordinate at m at whose intersection n with the curve draw a radial line On, making some angle θ with the vertical, and the abscissa nk. Now Om, or s, $= r \sin \theta$; and the length of curve En is proportional to the angle θ. Hence the *linear arc En is proportional to the time occupied by the body* in describing the distance $s = Om$; that is, if n be regarded as a moving point, confined to the circumference of the circle, and always in the vertical through m, its motion being thus controlled by the harmonic motion of m, *the velocity of n in its circular path is constant* (equal linear arcs in equal times). The constant velocity of n must be equal to the initial velocity

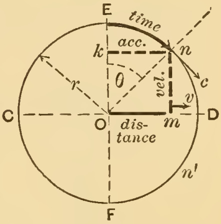


FIG. 54.

of m, i.e., c, as the latter leaves the origin, O; for at O the two points are both moving horizontally.

Conversely, therefore, if a point n move in the circumference of a circle with a constant velocity c (continuously in one direction), the foot of its ordinate, i.e., the point m, moves with harmonic motion along the horizontal diameter; this is the proof called for in the line below Fig. 64 of M. of E. The θ of Fig. 54 is $t\sqrt{a}$ of the formulæ on p. 59, M. of E.

Since the acceleration (as also accelerating force) in harmonic motion is proportional to the *displacement*, the length of the line Om, or kn, represents the acceleration at any instant, while the length, mn, of the ordinate is proportional to the velocity of the body or material point, m (since by eq. (4) of p. 59, M. of E., that velocity varies as the cosine of θ).

Of course, in obtaining values, from the drawing, of these variables v, p, and t, for different positions of the body m, due regard must be paid to the scales on which the lengths marked in the figure represent these variables, none of which is a linear quantity.

46. Numerical Example of Harmonic Motion (using the foot, pound, and second).—With the apparatus and notation of p. 58, M. of E., suppose that by previous experiment with the elastic cords we find that a tension T_1 of 4 lbs. is required in either of them to maintain an elongation of 3 in., i.e., $\frac{1}{4}$ of a foot; then for any elongation s (or "displacement" of the body during its motion) the tension (and retarding force) is $T = T_1 s \div s_1 = 16s$. Let the small block weigh 128.8 lbs., then its mass is $128.8 \times .0310 = 4 = M$; and hence the constant quantity called a is

$$a = (T_1 \div Ms_1) = 16 \div M; \quad \text{i.e.,} \quad a = 4.$$

Let an initial velocity of $c = 4$ ft. per sec., from left to right, be given to the block at O; required the extreme distance attained by the block from O (i.e., the semi-amplitude," $= r$), and the time occupied in describing the semi-amplitude (i.e., required the time of a quarter-period).

From p. 59, M. of E., $r = c \div \sqrt{a}$; $\therefore r = 4 \div \sqrt{4} = 2$ ft.; while for a quarter-period, or half-oscillation, we have the time $\frac{1}{2}\pi \div \sqrt{a} = \frac{1}{4}\pi = 0.7854$ sec.

Hence in the diagram of Fig. 54 we make $OD = r = 2$ ft. Then, since the linear arc ED represents 0.7854 seconds, the time represented by En for any position m of the block will be on a scale of $\frac{1}{4}$ sec. to each foot of En (since $0.7854 \div (\frac{1}{2}\pi \times 2 \text{ ft.}) = \frac{1}{4}$). Since OE, or 2 ft., represents the velocity of m on leaving O, i.e., 4 ft. per second, mn represents the velocity at m on a scale of 2 ft. per second to each foot of mn. Similarly, since the acceleration OD of m at D (from $p = -as$) is $-(4 \times 2) = -8$ ft. per (sec.)², that at m will be kn on a scale of 4 to each foot of kn.

It must be noted that after m reaches D and begins to return toward the left, the point n in the curve will have passed *below* the horizontal through D; that is, the time is now greater than ED, and the velocity has changed sign. When n passes underneath O the acceleration and the displacement change sign; and so on indefinitely. (Tension in cord at $D = ?$)

47. The Ballistic Pendulum.—This old-fashioned apparatus for determining the velocity of a cannon-ball consisted of a heavy body B, of mass M_2, suspended, by a rod or rods, from a horizontal hinge on a fixed support. A cavity in its side is partly filled with clay, so as to make the impact *inelastic*. This body being initially at rest (see Fig. 55), the ball is shot horizontally into the cavity, in which it adheres. As a result of the impact the centre of gravity of the combined masses receives a velocity C, with which it begins its ascent along the circular arc Bm, finally reaching a vertical height H above its initial position before coming to rest (for an instant). H being measured, or computed from the observed angle β, we have $C = +\sqrt{2gH}$ (see foot of p. 80, M. of E.). The known mass of the ball being M_1 and its unknown velocity just before impact $+c_1$, eq. (4) of p. 65, M. of E., enables us to write

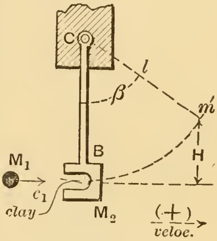


FIG. 55.

$$+\sqrt{2gH} = \frac{+M_1c_1 + 0}{M_1 + M_2}; \quad \text{i.e.,} \quad c_1 = \left(\frac{M_1 + M_2}{M_1}\right)\sqrt{2gH}.$$

(In this theory the suspended mass has been treated as a material point, and the result is only approximate. The longer and lighter the suspension-rod and the smaller the dimensions of the suspended body the more accurate the results. The strictly correct theory is rather complicated.)

48. Apparatus for the Determination of the "Coefficient of Restitution."—It will be noted, by an examination of eqs. (2), (3), (5), and (6), on pp. 64 and 65, M. of E., that the coefficient of restitution, e, may be defined as the ratio of the loss of momentum of the first body in the second period of the impact (period of expansion or restitution) to the momentum lost by it in the first period (compression); and similarly for the second body, except that for it there is a gain of momentum in both periods, instead of a loss; hence

$$e = \frac{M_1(C - V_1)}{M_1(c_1 - C)}; \quad \text{and also} \quad e = \frac{M_2(V_2 - C)}{M_2(C - c_2)}.$$

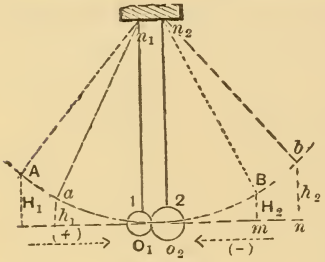


FIG. 56.

Fig. 56 shows the apparatus. The two balls, of same substance, are suspended by cords so that they can vibrate in the same vertical plane. When hanging at rest they barely touch, without pressure, with centres at the same level. Being allowed to swing simultaneously from rest at a and b respectively, their impact takes place at O (or very nearly so), their velocities just before impact being $c_1 = +\sqrt{2gh_1}$ and $c_2 = -\sqrt{2gh_2}$, respectively (see foot of p. 80, M. of E.), where h_1 and h_2 are the vertical heights fallen through (each ball having underneath it a graduated arc, so that each of these heights can be computed from the observed angle and known length of cord). Supposing each ball to rebound on its own side of O and the heights reached, H_1 and H_2, to be noted or computed, their velocities at O immediately after impact must have been $V_1 = -\sqrt{2gH_1}$ and $V_2 = +\sqrt{2gH_2}$ respectively, A and B being the points reached at the ends of the rebounds. Knowing, then, h_1 and h_2, M_1 and M_2, H_1 and H_2, we compute e from either eq. (7) or (8) of p. 65, M. of E.

Of course e is always less than unity, and due regard must be paid to the signs of the velocities in making the computations.

49. Action of a Spring between Two Balls (Fig. 57).—In the second period of a direct central impact (period of expansion, or restitution) the distance between the centres of the two masses is increasing, the front body being subjected to a forward pressure precisely equal to that which at the same instant is retarding the hinder body. Hence the gain in momentum of the front body during this period is equal to the loss in momentum of the other body; i.e. (see eqs. (5) and (6), p. 65 of M. of E.), $M_2(V_2 - C) = M_1(C - V_1)$.

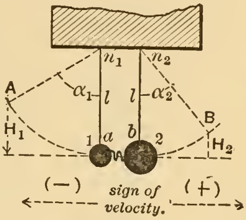


FIG. 57.

This is precisely what takes place when a spring in a state of compression (the ends tied together by a cord) is placed endwise between the two suspended balls of the apparatus of Fig. 57, and the cord is then severed without violence (burning is best). The spring in regaining its natural length exerts equal horizontal pressures in opposite directions at any instant against the two masses. This pressure is variable, but at any instant has the same value for one mass as for the other. That is, this phenomenon may be looked upon as a case of impact where the first period is lacking, the period of restitution occupying the whole time of impact. In this case M_1 and M_2 are at rest just before impact, that is, $C = 0$; while their velocities after the expansion of the spring are $-\sqrt{2gH_1}$ and $+\sqrt{2gH_2}$ respectively (if H_1 and H_2 are the respective vertical heights reached by the balls as the result of the action of the spring). Hence, from the above equation, $M_2(+\sqrt{2gH_2} - 0) = M_1(0 - [-\sqrt{2gH_1}])$; i.e., the velocities immediately after the action of the spring are inversely proportional to the masses.

The same method in another form consists in saying that from the principle of the conservation of momentum the total momentum before impact, viz., 0, here, is equal to that after impact, which is $M_1V_1 + M_2V_2$, or $M_1(-\sqrt{2gH_1}) + M_2(+\sqrt{2gH_2})$. Equating the latter expression to the zero momentum first mentioned, we have $M_1\sqrt{2gH_1} = M_2\sqrt{2gH_2}$, as before. This result is verified by the apparatus.

50. The Cannon as Pendulum.—As the converse of the Ballistic Pendulum, the cannon itself may be suspended in a horizontal position from a pivot C, Fig. 58. With the ballistic pendulum the impact was inelastic, i.e., it consisted of a period of compression only, that of restitution being lacking. Here, however, the expansion of the hot gases generated when the powder is ignited acts like the spring in the preceding case, causing a forward pressure at any instant against the ball and an equal and simultaneous backward pressure against the cannon; i.e., the impact is one having no period of compression but only one of restitution. The velocity V_2 of the ball leaving the muzzle may therefore be inferred from a knowledge of the two masses concerned and the height of recoil H_1 (A being the point where the centre of gravity of the gun comes to rest for an instant). Putting the total momentum before impact, which is zero (cannon at rest, in lowest position), equal to that immediately after, the cannon having then just begun its motion from D toward A with a velocity at D of $V_1 = -\sqrt{2gH_1}$, we have

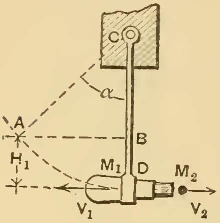


FIG 58.

$$0 = M_2V_2 + M_1(-\sqrt{2gH_1}); \quad \text{or} \quad V_2 = \frac{M_1}{M_2}\sqrt{2gH_1}.$$

As before, we here treat the masses as material points.

51. Simple Circular Pendulum of Small Amplitude (Fig. 59).—B is a material point suspended from a fixed point C by an imponderable and *inextensible* cord. Being allowed to sink from initial rest at the point A (cord taut), it follows the circular arc ABO with increasing velocity. At any point B between A and O its velocity (which of course is tangent to the curve) is

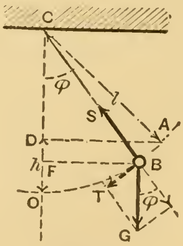


FIG. 59.

$$v = \sqrt{2g \times \text{vertical height } DF},$$

(see foot of p. 80, M. of E.), and the forces acting on it are its weight G, vertically downward, and the tension S, in the cord. This tension increases as the body approaches O, since Σ (norm.

comps.) must $= Mv^2 \div$ rad. of curv., [eq. (5), p. 76, M. of E.], i.e.,

$$S - G\cos\phi = \frac{G}{g}\cdot\frac{v^2}{l}; \quad \text{whence} \quad S = G\left[\cos\phi + \frac{v^2}{gl}\right]. \quad . \quad (1)$$

At O the tension is greatest and $= G\left[1 + \frac{2h}{l}\right]$. If the starting point A is taken high enough, the cord may be broken before O is reached. To find the tangential acceleration p_t, or rate at which the velocity is increasing, we note that from eq. (5), p. 76, M. of E., Σ (tang. comps.) must $= Mp_t$, whence

$$Mp_t = G\sin\phi + S \times 0; \quad \text{or} \quad p_t = g\sin\phi. \quad . \quad (1)$$

This can be written $p_t = \frac{g}{l}\cdot l\sin\phi = \frac{g}{l}\cdot\overline{BF}$. . . . . (2)

But if the angle ACO is very small (not over 5°, say), $\overline{BF}$ may be put = linear arc $\overline{OB}$; and as O is the middle of the oscillation, and BO is the distance or displacement of the body from O at any instant, (2) may be stated in the form : *the acceleration is proportional to the displacement*; so that the motion is (very nearly) harmonic (see p. 58, M. of E.). To find the time of passage from A to O on this basis (half-oscillation), note that on p. 59, etc., M. of E., the time of a half-oscillation is $\frac{\pi}{2}\cdot\frac{1}{\sqrt{a}}$, where a is the quantity which multiplied by the displacement s gives the acceleration. Hence from eq. (2) above, the "a" of the present harmonic motion is $g \div l$. Hence the time of a whole oscillation from A to the corresponding extreme point on the other side of O is

$$t' = \pi\sqrt{\frac{l}{g}}.$$

The duration of the oscillation is independent of the amplitude (with above limitations). (For a large amplitude see foot of p. 81, M. of E.)

With an extensible cord, elastic or inelastic, the results would be quite different. In such a case the relation $v = \sqrt{2g\,.\,\overline{DF}}$ for the velocity at any point B would not be true, since the tension S is not perpendicular to the velocity when the cord is elongating.

CHAPTER IV.

Numerical Examples in Statics of a Rigid Body and Dynamics of a Material Point.

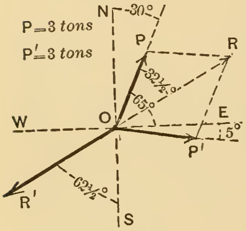


Fig. 60.

52. Example 1. Anti-resultant of Two Forces (Fig. 60).—Two forces in a horizontal plane, P and P', are respectively 3 tons (*short* tons, of 2000 lbs. each), and 6000 lbs. P is directed North 30° East; and P', South 85° East. The angle between them is therefore 65°. Required the amount and position of R', their anti-resultant (i.e., the force that will balance them). R' must be equal and opposite to the (ideal) resultant, R, of the two forces. Adopting the short ton as a unit for forces, we note that P' and P are equal, each being 3 tons. Hence the parallelogram of forces formed on them as sides is a *rhombus*, and R bisects the angle between ; whence we may write

$$R = 2P \cos 32\tfrac{1}{2}° = 2 \times 3 \times 0.8434 = 5.06 \text{ tons.}$$

Therefore an anti-resultant of 5.06 tons, directed South 62½° West must be provided, if the two given forces are to be balanced by a single force.

53. Example 2. Resultant Couple of a Number of Couples.—Six couples acting on a rigid body and in the *same plane* (a vertical plane) have the following moments, respectively, as seen from the west side of the plane:

+ 30 ft.-lbs. ; + 196 ft.-oz. ; + 0.48 inch-tons ;
— 0.08 ft.-tons ; — 160 ft.-lbs.; and — 1300 inch-lbs.

Required the moment of their resultant couple.

Changing the form of these various moments to what they would have been if all the various forces had been expressed in lbs. and the arms in feet (i.e., expressing them all in ft.-lbs.), and adding those of the same sign, we have

$$
\begin{array}{lll}
 & +(30+12.25+80)\text{ ft.-lbs.} & =+122.25\text{ ft.-lbs.};\\
\text{and} & -(160+160+108.33)\text{ ft.-lbs.} & =\underline{-428.33} \quad \text{“}\\
 & & -306.08 \quad \text{“}
\end{array}
$$

Since the algebraic sum of the moments is -306.08 ft.-lbs. (the couples being in the *same plane* (see § 34, M. of E.), this is the moment of their (ideal) resultant couple. To hold the given couple in equilibrium, therefore, a single couple (their anti-resultant) having a moment of $+306.08$ ft.-lbs. must be applied to the body to preserve equilibrium. That is, if a seventh couple in which (for example) each force is 153.04 lbs., with an arm of 2 ft., and appearing counter-clockwise seen from the west, be applied to the body, the seven couples will balance.

If the given couples were not all in the same plane, or in parallel planes, we should have to make use of the results of § 33, M. of E.

54. Example 3. Centre of Gravity of Trapezoid with Semicircle Cut Out (homogeneous thin plate of uniform thickness—Fig. 61).—The diameter of the semicircle lies on the lower base OK of the trapezoid, H is the centre of gravity of the semicircle (so that $HD = 4r \div 3\pi$), E that of the complete trapezoid, and E' that of the actual plate. See figure for given numerical dimensions and for notation of co-ordinates of centres of gravity of actual plate and complete trapezoid, denoting those of the semicircle by $\bar{x}_2$ and $\bar{y}_2$. Required $\bar{x}_1$ and $\bar{y}_1$, using the *inch* as linear unit. The area of the trapezoid is $F = \frac{1}{2} \times 6 \times (10+16) = 78$ sq. in.; that of the semicircle is $F_2 = \frac{1}{2}\pi(2)^2 = 6.28$ sq. in.; so that the area of the actual plate is their difference, F_1, $= 71.71$ sq. in. Also, for the point H we have $\bar{y}_2 = 4 \times 2 \div 3\pi = 0.85$ in.; and $\overline{OD} = \bar{x}_2$

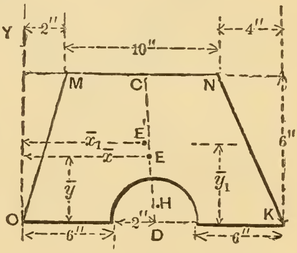


FIG. 61.

$= 8$ in.; while (see p. 23, M. of E.) $\bar{y} = \frac{6}{3} \cdot \frac{16 + 20}{16 + 10} = 2.77$ in.; and $\bar{x} = \overline{OD} - \frac{1}{6} \times 2.77 = 7.54$ in.

(C bisects the base MN; conceive perpendiculars let fall from C and E upon OK, and use the similar triangles so formed in obtaining the value of $\bar{x}$.)

From eq. (3), p. 19, M. of E., since the trapezoid is made up of the actual plate and the semicircle, we have

$$\bar{x} = (F_1\bar{x}_1 + F_2\bar{x}_2) \div (F_1 + F_2);$$

$$\therefore\ \bar{x}_1 = \frac{F\bar{x} - F_2\bar{x}_2}{F_1}; \quad \text{and} \quad \bar{y}_1 = \frac{F\bar{y} - F_2\bar{y}_2}{F_1}.$$

By substitution, therefore,

$\bar{x}_1 = [78 \times 7.54 - 6.28 \times 8] \div 71.71 \quad = 7.5$ in.; and

$\bar{y}_1 = [78 \times 2.77 - 6.28 \times 0.85] \div 71.71 = 2.94$ "

55. Example 4. Stability of Two Cylinders.—Fig. 62 gives an end view of two smooth and homogeneous circular cylinders of equal length ($= l'$ in feet) but of radii 1 in. and 3 in., respectively. A weighs 800 lbs. per cub. ft., B only 100 lbs. per cub. ft. A and B being placed, as shown, on two smooth planes at 45° with the horizontal, it is required to find whether this is a stable position, or if A will crowd B out of place. Call their total weights G_1 and G_2 for the present. Since $\overline{CD}$ is one half of $\overline{OC}$, the angle COD is 30°. First, the position being supposed stable, to find the pressure at point 2, we take A as a free body. The forces acting on it are three, shown at O in Fig. 63. P_1 is the pressure (or reaction) of the inclined plane against A at point 1, P_2 is the pressure from the other cylinder, and G_1 the weight of this cylinder, A. These are all directed through O (smooth surfaces), the angles being as shown. For equilibrium P_2 must be equal and opposite to OK, the (ideal) resultant of G_1 and P_1. In triangle OP_1K, $OK : G_1 :: \sin 45° : \sin 60°$; $\therefore\ \overline{OK} = G_1[\sqrt{2} \div \sqrt{3}] = P_2$.

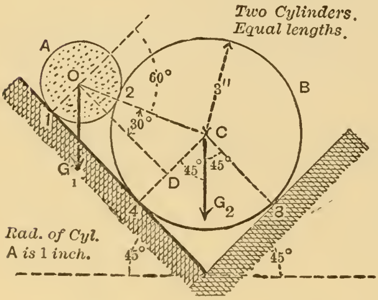


FIG. 62.

In Fig. 63 we have acting through C the four forces acting on the large cylinder B (on supposition of equilibrium). G_2 is its weight, P_2 is the pressure of the other cylinder, P_3 the pressure at point 3 of the inclined plane on the right, P_4 of that on the left.

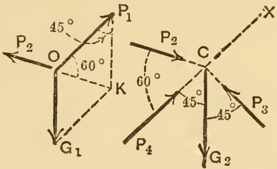


FIG. 63.

If now the resultant of P_2 and G_2 were found to pass above the point 3, instability would be proved; since to occasion pressure at point 4 that resultant should evidently pass below 3. Or, which amounts to the same thing, assuming equilibrium at C and with P_4 as drawn, if we compute the value of P_4 by putting Σ (compons. ⌉ to P_3) = 0, and a negative result is obtained, instability is proved; and *vice versa*. Hence we write

$$P_4 = G_2 \cos 45° - P_2 \cos 60° ; \text{ whence } P_4 = \frac{G_2}{\sqrt{2}} - \frac{G_1}{\sqrt{6}} . \quad . \ (1)$$

Now $G_1 = [\pi(\frac{1}{12})^2 \times 800 l']$ lbs.; and $G_2 = [\pi(\frac{3}{12})^2 \times 100 l']$ lbs., on substituting which in (1) we obtain

$$P_4 = [\tfrac{1}{144}\pi l'(+309.8)] \text{ lbs.};$$

which is positive, and thus verifies the supposition of equilibrium.

56. Example 5. Toggle-joint (Fig. 64).—The two straight links, of equal length, are pivoted to the two blocks, as indicated. A horizontal pull of 80 lbs., making equal angles with the two links, is exerted on the horizontal pin of the joint B. What pressures are thereby induced (for given position of parts) on the surfaces E and F (horizontal and vertical)? (Those on E' and F' will be the same, respectively.) Each link is evidently a straight two-force piece and hence under a compressive stress along its axis; call this stress P'. The free body in Fig. 65 enables us to find P' from Σ (hor. comps.) = 0; i.e., $2P' \cos \alpha = P$; or $P' = \dfrac{P}{2 \cos \alpha}$.

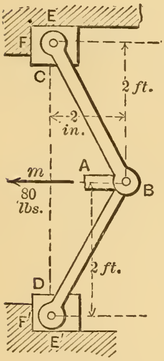


FIG. 64.

Fig. 66 shows the upper block free, from which by $\Sigma Y = 0$, $P_E = P' \sin \alpha$, and from hor. comps. $P_F = P' \cos \alpha$.

$\therefore P_E = \frac{1}{2} P \tan \alpha = \frac{80}{2} \cdot \frac{24}{2} = 480$ lbs.; and $P_F = \frac{1}{2} P = 40$ lbs.

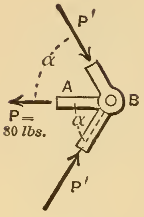


FIG. 65.

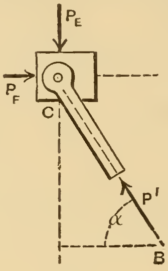


FIG. 66.

If, in Fig. 65, P were resolved into components along the axes of the two links, each such component would be the equal and opposite of the corresponding P'. Evidently, if α approached a right angle, P', and also P_E, would *increase without limit.*

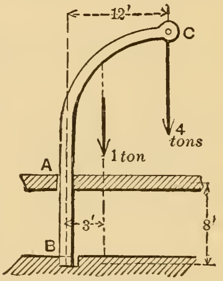


FIG. 67.

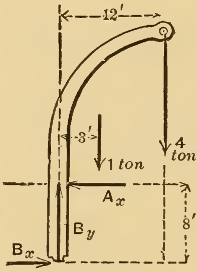


FIG. 68.

57. Example 6. Simple Crane.—The *simple crane* in Fig. 67 carries a load of 4 tons at C, 12 ft. from the axis of the vertical shaft, while its own weight is 1 ton, the centre of gravity being 3 ft. from the shaft. The socket at B is shallow so that lateral support is provided at A. Required the pressure at A and the horizontal and vertical pressures from side and bottom of socket B. The crane being considered free in Fig. 68, and the reacting pressures being put in, as shown, we have, from Σ (moms. about B) $= 0$ (foot and ton),

$$A_x \times 8 - 4 \times 12 - 1 \times 3 = 0; \quad \text{whence} \quad A_x = 6.37 \text{ tons}.$$

From Σ (vert. compons.), $B_y - 5 = 0$; or $B_y = 5$ tons.

Σ (hor. compons.), $B_x - A_x = 0$; or $B_x = 6.37$ tons.

58. Example 7 Door and Long Hinge-rod.—The door weighs 200 lbs., and is supported in a vertical plane in the manner indicated in Fig. 69. The continuous vertical rod ABC is considered without weight, and from the nature of the mode of support of the door receives horizontal pressures at each of the points A, B, C', and C. Required the values (A, B, C', and C) of these forces; also the vertical pressure C'' between the projecting shoulders C and C'.

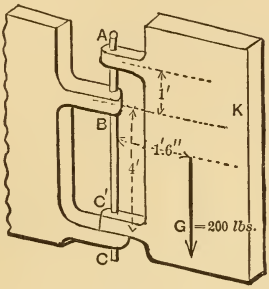


FIG. 69.

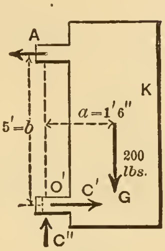


FIG. 70.

Fig. 70 shows the door free, and, by moments about O', using the pound and foot,

$$Ga = Ab; \text{ or } A = (200 \times 1.5) \div 5 = 60 \text{ lbs.}$$

Σ (vert. comps.) gives

$$C'' = G = 200 \text{ lbs.}; \text{ while from}$$

Σ (hor. compons.), $-A + C' = 0$, whence

$$C' = A = 60 \text{ lbs.}$$

The rod as a free body is shown in Fig. 71 and enables us to find the pressures B and C, now that A and C' are known. By moments about o, we have

$$B \times 4 - 60 \times 5 = 0; \text{ whence } B = 75 \text{ lbs.}$$

In putting Σ (hor. compons.) $= 0$, since $C' = A$ from Fig. 70, the summation reduces to $C - B = 0$, i.e., $C = 75$ lbs.

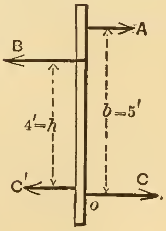


FIG. 71.

The hinge-rod, therefore, is seen to be under the action of two couples of equal and opposite moments; one consisting of A and C', the other of B and C, while the sill C, Fig. 69, receives a vertical pressure equal to the weight of the door.

59. Example 8. Shear-legs (Fig. 72).—The weight G of

2000 lbs. is supported by the two straight links in a vertical plane as shown, with given dimensions and weights. Required the pressures produced on the hinge-pins at A and B. If the links had no weight they would be straight two-force pieces, CB being subject to a compression and AC to tension; and the hinge-pin pressures at A and B would be equal to these forces, the action-lines of the latter being the axes of the pieces, respectively. In that case a simple solution would consist in resolving the force of 2000 lbs. by a parallelogram into two components, one along CB, the other along AC prolonged, and these components would be found to be 3367 and 1813 lbs., respectively; the former being the compression in CB and the latter the tension in CA. But the weights of the links are considerable and are to be considered; hence the links are not two-force pieces and must not be conceived to be cut in forming any free body. The hinge-pressures at A and B are replaced by their horizontal and vertical components, as shown, A_x and A_y, B_x and B_y; these four are the unknown quantities required.

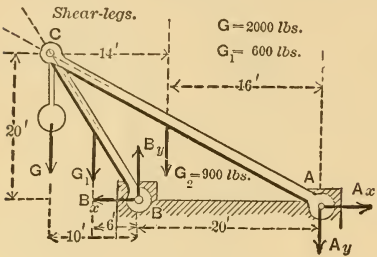


FIG. 72.

The figure shows all the forces acting on a free body consisting of the two links and the 2000-lb. weight. By taking moments about A we exclude three of the unknown quantities and obtain

$$+2000 \times 30 + 600 \times 26 + 900 \times 16 - B_y \times 20 = 0;$$

$$\text{or } B_y = 4500 \text{ lbs.};$$

while by moments about B,

$$2000 \times 10 + 600 \times 6 - 900 \times 4 - A_y \times 20 = 0;$$

$$\text{or } A_y = 1000 \text{ lbs.}$$

Now conceive the link CA alone to constitute a free body, the forces acting being A_x, A_y, G_2, and the pressure of the hinge-pin at C against this link. The action-line of this last force is not known, but the moment-sum about C excludes the force and gives

$$+A_x \times 20 - 1000 \times 30 - 900 \times 14 = 0\text{; whence } A_x = 2130 \text{ lbs.}$$

Since in the first free body B_x must $= A_x$ (from Σ hor. comps. $= 0$), we have also $B_x = 2130$ lbs. and can now compute the actual oblique hinge-pressures A and B, at A and B respectively. From

$$A = \sqrt{A_x^2 + A_y^2}, \quad \text{and} \quad B = \sqrt{B_x^2 + B_y^2},$$

we have finally

$$A = 2354 \text{ lbs.}, \qquad \text{and} \quad B = 4978 \text{ lbs.}$$

From $\tan^{-1}(A_y \div A_x)$ we find that A makes a smaller angle with the horizontal than the link CA; and similarly, that of B is greater than that of link CB.

60. Example 9. Roof Truss with Loads and Wind Pressures (Fig. 73).—Here the half-weight of each piece is supposed to be carried directly on the pin of the corresponding joint, so that each link or member will be considered as a straight *two-force* piece and hence in simple compression or tension along its axis. In obtaining free bodies, therefore, any piece or pieces may be conceived to be cut and the stress inserted. The load given at each joint includes the half-weights of all the pieces meeting there. For all distances and angles needed see figure. The wind is supposed to blow from the left, its pressure (4 tons) on the left

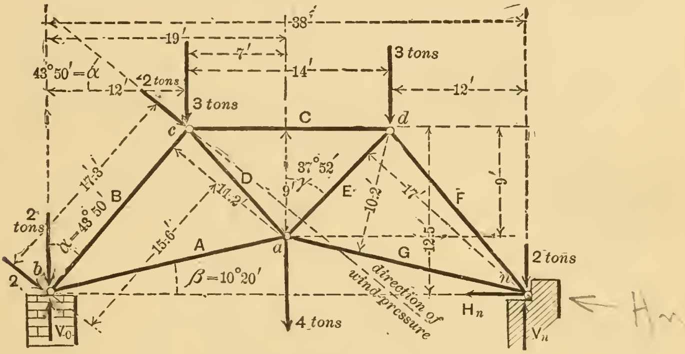


Fig. 73.

slope of the roof being normal to the same, half borne at each joint, b and c. Resistance to horizontal displacement is supposed to be provided at the right support, alone; the other extremity of the truss being on rollers, so that the reaction there is vertical; hence at the right we have two reactions to deal with, horizontal and vertical; i.e., H_n and V_n.

Required the three supporting forces, H_n, V_n, and V_0; and also the stresses A, C, D, and E (and their character), in the pieces, A, C, D, and E.

Fig. 73 shows the whole truss as a free body. By moments about joint b, adopting the foot and ton as units, we have

$$+V_n \times 38 - 2 \times 38 - 3 \times 12 - 3 \times 26 - 4 \times 19 - 2 \times 17.3 = 0;$$

whence $V_n = 7.91$ tons. From Σ (vert. comps.) $= 0$,

$$V_0 + V_n - 2 - 3 - 3 - 4 - 2 - 2 \sin \alpha - 2 \sin \alpha = 0;$$

and hence $V_0 = 8.88$ tons; while Σ (hor. comps.) $= 0$ gives

$$+2 \cos \alpha + 2 \cos \alpha - H_n = 0; \quad \text{or} \quad H_n = 2.86 \text{ tons.}$$

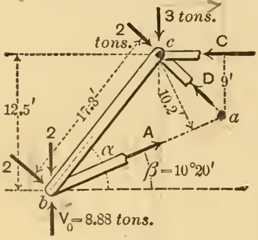


FIG. 74.

Next, considering free the portion of the truss on the left of a plane cutting pieces A, D, and C, we have Fig. 74, in which, for the present, we assume A to be tension and C and D compression. Σ (moms.) about point c (*intersection of C and D*) gives

$$A \times 10.2 + 2 \times 12 + 2 \times 17.3 - 8.88 \times 12 = 0;$$

whence $A = +4.7$ tons, and this being positive, the assumption of tension is confirmed. Σ (moms.) about a, similarly, gives

$$C \times 9 + 3 \times 7 - 2 \times 1.7 + 2 \times 19 + 2 \times 15.6 - 8.88 \times 19 = 0;$$

or, $C = +9.1$ tons, and is therefore compression. Although the same free body would serve, let us determine stress D from another free body, that in Fig. 75, showing the remainder of the truss. Assume D compression.

From Σ (vert. comps.) $= 0$ we have

$$+7.91 - 3 - 2 - 4 - A \sin \beta - D \cos \gamma = 0;$$

or, $D = -2.45$ tons. The negative sign showing the assumption of compression to be incorrect, D is 2.45 tons tension.

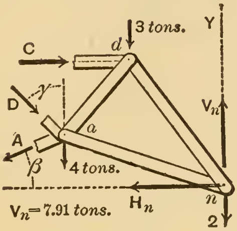


FIG. 75.

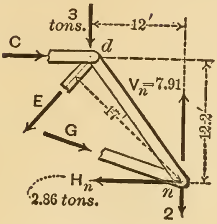


FIG. 76.

Again, from the free body in Fig. 76, taking moments about n, having assumed E to be tension, we have

$$+E \times 17 + 3 \times 12 - 9.1 \times 12.5 = 0;$$

i.e., $E = +4.56$ tons, and is tension.

From the same free body the stress in G is easily found.

61. Remark.—The foregoing examples of this chapter have all involved the equilibrium of rigid bodies, each under a system of forces in a plane. Those remaining to be given, however, deal with *moving* material points, or bodies small enough to be so considered (dynamics of a material point); and the concurrent forces in each case acting on the body when considered free, do not form a balanced system (unless the motion is rectilinear and of constant velocity), so that ΣX and ΣY are not $= 0$ necessarily. For example, if the path is a *straight line* which is taken as the axis X, then $\Sigma X =$ *mass* $\times$ *accel.*; while Σ (*comps.* ┐ *to the path*) $= 0$, as if the forces were balanced. But if the path is a *curve*, then at each point Σ (*comps. along the tangent*) $=$ *mass* $\times$ *tan. acc.* and Σ (*comps. along the direction of the normal*) $=$ *mass* $\times$ *square of veloc.* $\div$ *radius of curvature.* In numerical substitution the student is very apt to forget that if g, the acceleration of gravity, be denoted by the number 32.2, *times* must be expressed in *seconds* and *distances* in *feet.* (The expression for the mass of a body always involves the quantity g.)

62. Example 10. Train Resistance.—If the frictional resistance of a certain 200-ton railroad train be assumed to be equivalent to a backward force of 12 lbs. per ton applied directly to the car-frames at any ordinary speed, in what distance on a level track will the train be stopped if moving initially at a velocity of 40 miles per hour? (There are no brakes on, nor any locomotive; the resistance being due to the rubbing of the journals in their boxes and the unevenness and compressibility of the track and wheel-treads (rolling resistance)).

Ditto: if the train is on an up-grade of 26.4 ft. to the mile?

Taking the axis $+X$ in the direction of motion, we note that the accelerating force is — 2400 lbs.; i.e., that the sum of the comps. along the path is — 2400 lbs. The mass, in the ft.-lb.-sec. system of units, is $= G \div g = 400{,}000 \div 32.2$. Now $\Sigma X = Mp = mass \times acc.$, and therefore the $acc. = p = \Sigma X \div M = -0.193$. The accelerating force being constant, the acceleration is constant and hence the motion is uniformly accelerated (retarded here), and the eq. (3) of p. 54, M. of E., is applicable, viz, $distance = s = (v^2 - c^2) \div 2p$. The initial velocity $= c = 40$ miles per hour, $= 58.6$ ft. per sec., while v is to be zero. Hence

$$s = \frac{0^2 - (58.6)^2}{2 \times (-0.193)} = 8905 \text{ ft.}$$

On the up-grade, the path, or axis X, is inclined upward at an angle α with the horizontal (whose tangent is $26.4 \div 5280 = \frac{1}{200}$ and is practically $= \sin \alpha$) and, besides the — 2400 lbs., the X component of G, viz., $-G \sin \alpha = \frac{1}{200}$ of 400,000 lbs. $=$ — 2000 lbs., acts to retard the motion.

$$\therefore\ \Sigma X = -4400 \text{ lbs.} \quad \text{and} \quad p = -0.354 \text{ ft. per sec. per sec.}$$

$$\therefore\ s = [0^2 - (58.6)^2] \div [2 \times (-0.354)] = 4858 \text{ ft.}$$

63. Example 11. Inclined Plane, Two Weights, and Cord (Fig. 77).—The cord connecting the two weights is very light and inextensible, and friction and mass of the pulley are neglected. (By neglecting the *mass* or inertia of the pulley we mean that, notwithstanding the fact that its rotary motion is accelerated by the cord, the tension in the cord is the same at any instant where it leaves, as where it winds upon, the pulley-rim.) The two blocks being at rest in the position shown (cord taut, with a temporary support under B), the support is suddenly removed; required the distance s_2 through which B then

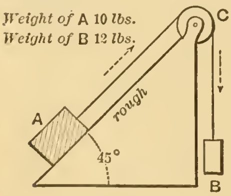


Fig. 77.

sinks in the first two seconds of time ($=t_2$), and also the tension in the cord during the motion, if the body A encounters a frictional resistance, on the inclined plane, always equal to $\frac{3}{10}$ of the normal pressure on the plane.

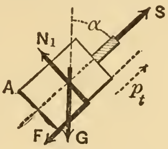


FIG. 78.

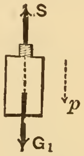


FIG. 79.

Consider A free at any instant of the motion (Fig. 78). Call its acceleration p_t. The forces acting on it are its weight G, S the tension in the cord, the friction F, and the normal pressure N_1 from the inclined plane. (That is, the resultant of N_1 and F is the resultant action of the plane on body A, which resultant action is evidently not normal to the plane, which it could not be unless the bodies were smooth.) Although the path of A is straight, consider it as a particular case of a curve; then (see § 61 above)

$$\Sigma\,(\textit{tang. comps.}) \quad = S - F - G\cos\alpha = \frac{G}{g}p_t; \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma\,(\textit{normal comps.}) = \frac{Mv^2}{r}; \quad \text{or} \quad N_1 - G\sin\alpha = \frac{Mv^2}{\infty} = 0. \quad (2)$$

From (2) we find the value of N_1; and hence

$$F = \tfrac{3}{10}N_1 = \tfrac{3}{10}G\sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At this same instant (which is *any* instant of the motion), consider B free in Fig. 79. There are only two forces acting on it: its weight G_1, and the upward tension S_1 in this part of the cord. B is sinking with some acceleration p.

From Σ (*downward comps.*) = *mass* × *acc.*, we have

$$G_1 - S_1 = \frac{G_1}{g}p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But (from above remark on mass of pulley, etc.) we know that $S_1 = S$; and since the cord is taut and *does not stretch*, p must $= p_t$. (Let the student devise a strict proof of this.)

Hence by elimination between the four equations we obtain

$$p = p_t = \frac{G_1 - F - G\cos\alpha}{G + G_1}\cdot g,$$

which is *constant* and hence the motion is *uniformly accelerated* and the equations of § 56, M. of E., hold good, among which is $s = \frac{1}{2}pt^2$ when the initial velocity is zero.

Passing to numbers, in the ft.-lb.-sec. system, we have

$$p = \frac{12 - 10 \times 0.707\,(0.3 + 1)}{10 + 12} \cdot (32.2) = 4.11 \text{ ft. per. sec. per. sec.},$$

and hence from (4),

$$\text{tension} = S = 12\left[1 - \frac{4.11}{32.2}\right] = 10.47 \text{ lbs.};$$

while $s_2 = \frac{1}{2}pt_2^2 = \frac{1}{2} \times 4.11 \times (2)^2 = 8.22$ feet = distance described in the first two seconds (the velocity at end of which $= v_2 = pt_2 = 8.22$ ft. per sec.).

It is seen that the weight B sinks with about one eighth the acceleration of a free fall.

64. Example 12. Free Fall.—A stone, allowed to descend freely and vertically, from rest, occupies $\frac{17}{4}$ of a second (17 watch-ticks, say) in falling through the height of a cliff; required this height. From eq. (2), p. 51, M. of E., we have $s = ct + \frac{gt^2}{2}$. In the present case $c = 0$, and hence the height required $= 0 + 32.2 \times \frac{1}{2} \times (\frac{17}{4})^2 = 290.8$ ft.

On account of atmospheric resistance, which is neglected in the theory of p. 51, M. of E., and is variable (being nearly proportional to the square of the velocity for the same body), the actual height is smaller, the discrepancy depending on the shape, the specific gravity, and *absolute size* of the falling body. If the stone is round, and about one inch in diameter, the average resistance in the above case might be as much as one-quarter of its weight, so that we might write $\frac{3}{4}g$ instead of g for a rough approximation. (See p. 822, M. of E.) If it were two inches in diameter (same substance), its weight would be increased eight-fold, and the average resistance about quadrupled, and thus the latter might be about one eighth of the weight.

With smaller heights of fall the resistance is much smaller, not only absolutely but proportionally, on account of the smaller average velocity.

65. Example 13. Block on Circular Guide (Fig. 80).—The curved guide *ABD* is smooth and fixed, of the form of the quadrant of a circle with a horizontal tangent at *D*. The plane *DE* is rough. The block *G* weighs 20 lbs. and is to slide from rest at *A* down the circular guide. How far (i.e., distance $s_1 = ?$) will it slide on the rough plane *DE* before being brought to rest, if the latter offers a frictional resistance of 20 oz. (i.e., $1\frac{1}{4}$ lb.)? The radius of the curve in which the centre of *G* moves is 48 inches.

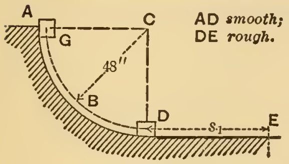


FIG. 80.

The velocity v_1 of *G* on its arrival at *D* is the same as if it fell freely through the corresponding vertical height $CD = 48$ in. $= 4$ ft. (the time of descent, however, is quite different); for the guide is both *fixed* and *smooth* (see p. 83 and also foot of p. 80, M. of E.). Adopt the foot, lb., and second.

$\therefore v_1 = \sqrt{2 \times 32.2 \times 4} = 16.05$ ft. per sec. For the motion on *DE*, v_1 is the initial velocity, and the motion is uniformly retarded (i.e., the acceleration is constant and negative) if we take the direction from *D* toward *E* as positive; since the only force-component along the path is -1.25 lbs., the gravity-force of 20 lbs. being ⌉ to the path. The acceleration is $p = force \div mass$, the mass being $= 20 \div 32.2 = 20 \times 0.0310 = 0.620$; $\therefore p = (-1.25) \div .62 = -2.012$ ft. per sec. per sec., and from eq. (3) of p. 54, M. of E. (in which, for present purposes, we put $s = s_1$, $v = 0$, $c = v_1$, and p as above), we have

$$s_1 = [-(16.05)^2] \div [2(-2.012)] = 64 \text{ ft.}$$

If we inquire the pressure P between the block and curved guide just before reaching *D*, we note that that pressure must not only support the weight of the body, but must also provide a proper deviating force $\frac{Mv_1^2}{r}$ to retain it on the curve;

$$\text{whence} \qquad P = 20 \text{ lbs.} + \left(\frac{0.62(16.05)^2}{4}\right) \text{lbs.} = 60 \text{ lbs.}$$

(See p. 83, M. of E.) The pressure on *DE* is only 20 lbs.

Again, suppose the block to start from rest at B, the angle BCD being 45°; find v_1 and s_1. (The acceleration on DE is the same as before.)

$$v_1 = \sqrt{2 \times 32.2 \times 4(1 - \cos 45^\circ)} = 8.68 \text{ ft. per sec.}$$

$$s_1 = (0^2 - v_1^2) \div [2(-2.012)] = 18.75 \text{ ft.}$$

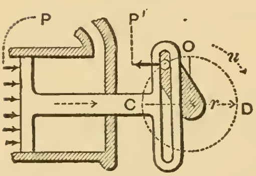


FIG. 81.

66. Example 14. Harmonic Motion of a Piston (Fig. 81).—On account of the great mass, and large radius, of the rim of a fly-wheel on the same shaft, the rotation of the crank is practically uniform; at least during any one turn, i.e., the crank-pin is considered to move with a *uniform* velocity in a circle. From the design of the piston and slot this body oscillates with harmonic motion in a horizontal path (see foot of p. 59, M. of E.); the left to right stroke alone is to be considered. The pressure called P is the total effective steam-pressure, i.e., the difference between the total pressure of the steam, now on the left of the piston, and the total atmospheric pressure on the right face. Friction on the guiding surfaces is neglected, and since the motion is horizontal the weight of the piston has no component along its path.

If P is constant throughout the whole stroke (left to right), and = 6000 lbs., and the crank turns uniformly at the rate of (u =) 200 revolutions per minute, r being = 8 inches; what must be the value of the pressure P' between crank-pin and the side of the slot just after the dead-point C is passed, i.e., at the beginning of the stroke? *Ditto*, when the crank-pin is 45° from C; and again, when it is at O, 90° from C? The weight of the piston and rod is 160 lbs.

Between C and O, P' is smaller than P and is a resistance, as regards the motion of the piston. If that motion were uniform the full amount of the 6000 lbs. would be felt at the pin, for in that case the acceleration would be zero and the horizontal forces, P and P', would be equal and oppositely directed; but from C to O the motion is *accelerated* (the piston has *no* velocity at C), so that a certain amount, $= mass \times accel.$, of the 6000 lbs. is absorbed in the "inertia" of the mass, so to speak, leaving only the remainder to be felt as a pressure P' at the crank-pin. This is expressed analytically by the relation Σ (hor. comps.) $= Mp$; i.e., $P - P' = Mp$.

Beyond O toward D the constraint of the mechanism is such as to bring about the gradual stopping of the piston, which at O has its greatest velocity ($= c$, $=$ to that which the pin has at *all times*), so that independently of the 6000 lbs. on the left the piston is, as it were, thrown against the crank-pin, the pressure produced against which at any instant from O to D must $= Mp$ over and above the 6000 lbs. due to steam action; i.e., $P' = 6000$ lbs. $+ Mp$ (where p is the *numerical* value of the acceleration, whose algebraic value is now negative), or, analytically, $P - P' = Mp$ where p has its algebraic value (negative when a number is inserted for it).

The linear velocity of the crank-pin is $= 2\pi ru$, $= 2 \times \frac{22}{7} \cdot \frac{2}{3} \cdot \frac{200}{60}$, $= 13.97$ ft. per sec. (using the ft., lb., and sec.). As the pin approaches and passes a dead-point the motion of the foot of the perpendicular let fall from it upon the horizontal diameter, along that diameter, is not only the motion of the piston, but is at this point normal to the path of the pin; hence the normal acceleration of the pin is the actual acceleration of the piston at a dead-point. Therefore $c^2 \div r$ (see eq. (4), p. 75; and also section § 75, and first line of p. 60, M. of E.) is the acceleration of the piston at C; and therefore just after passing the dead-point C we have

$$P' = P - M\frac{c^2}{r} = 6000 - \frac{160}{32} \cdot \frac{(13.97)^2}{\frac{2}{3}}$$

$$= 6000 - 1452 = 4548 \text{ lbs.}$$

The acceleration of the piston is proportional to the displacement, and hence at 45° from C we have

$$P' = P - M\left[\frac{c^2}{r} \cos 45°\right] = 6000 - 1452 \times .707 = 4937 \text{ lbs.}$$

At O, $p = 0$ and $P = P' = 6000$ lbs.

Beyond 0 at 45°

$$P' = 6000 + 1452 \times .707 = 7063 \text{ lbs.};$$

while just before reaching D

$$P' = 6000 + 1452 = 7452 \text{ lbs.}$$

On the return-stroke steam is admitted to the *right* of the piston and P' occurs on the other side of the slot, with same variation during the stroke as before.

67. Example 15. Conical Pendulum, or Simple Governor-ball (Fig. 82 [*a*]).—If the oblique part of the cord is to be 20 in. in length, what tangential velocity in a horizontal circle (centre at C) and what radius $= r$, for that circle, must be given to the material point G of 10 lbs. weight ($= G$) in order that motion in the circle shall be self-perpetuating and the weight G' of 40 lbs. may be sustained *at rest*?

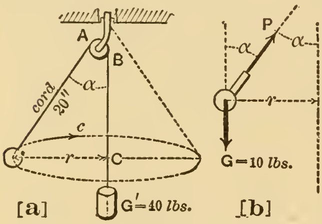


FIG. 82.

Fig. 82 [*b*] shows the moving weight as a free body, the only forces acting being a gravity-force of 10 lbs. and an oblique cord-tension, P, which by above conditions is to be 40 lbs. The motion of G being confined to a horizontal plane, it has no vertical acceleration; therefore Σ (vert. compons.) should balance, or $P \cos \alpha - G = 0$, whence $\cos \alpha = \frac{G}{P} = 0.25$, and α should $= 75° \ 31'$. Hence r should be made $=$ (20 in.) $\times \sin \alpha = 20 \times 0.968 = 19.36$ in. $= 1.613$ ft. Since the motion is to be in a curve, Σ (normal compons.) at any instant should $= M \times (vel.)^2 \div rad.$; i.e., $P \sin \alpha + 0 = \frac{G}{g} \cdot \frac{c^2}{r}$; and combining this with the $P \cos \alpha = G$ derived above we have $\tan \alpha = c^2 \div gr$, whence the required velocity must be $c = \sqrt{32.2 \times 1.613 \times 3.871} = 14.2$ ft. per sec. The proper radius being as above ($r = 1.613$ ft.), this implies rotation about C at the rate of

$$u = \frac{c}{2\pi r} = \frac{14.2}{2\pi \times 1.613} = 1.40 \text{ revolutions per second,}$$

or 84 per minute.

68. Example 16. The Weighted Governor or Conical Pendulum (Fig. 83).—The four inextensible cords, each 16 inches long, connect the three "material points," or small bodies, as shown; the two upper cords being attached to a fixed support at A. G_1 and G_1 are two balls of equal weight; the weight of each $= G_1 = 8$ lbs., while the block G_2 weighs 12 lbs. If now the balls are caused to rotate about the vertical axis ACB at slowly increasing rate (revolutions per minute), by pressing against them laterally with a vertical board whose plane contains the axis ACB, the angle α gradually increases and the block G_2 is lifted along the axis towards A. When the speed of rotation has reached any desired figure (rev. per min.) α has some corresponding value, and if the board is now removed α retains that value and the balls continue their motion (forever, if no friction) in the corresponding horizontal circle, sustaining the block G_2 at rest (at least its centre of mass is at rest) in some position B.

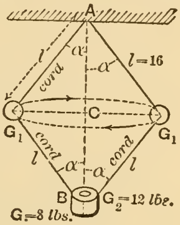


FIG. 83.

Required the distance AB, $= 2AC$, when a speed of rotation of 120 revs. per min. has been attained?

Let l denote the cord-length of 16 in., r the unknown radius of the horizontal circle, S_1 the tension induced in each of the upper cords, S_2 that in the lower, and c the unknown linear velocity of each ball. Let $u =$ the number of revolutions per unit time (so that $c = 2\pi ru$). Fig. 84 shows one of the balls as a free body. Since its vertical velocity is *always the same* (zero), i.e., its vertical acceleration $= 0$ (the motion being confined to a horizontal plane),

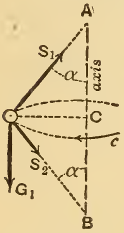


FIG. 84.

$$\Sigma\,(\text{vert. comps.}) = 0;\ \text{i.e., } S_1 \cos\alpha - S_2 \cos\alpha - G_1 = 0; \quad . \quad (1)$$

while on account of the curvilinear motion Σ (normal compons.) $= Mc^2 \div r$, or

$$S_1 \sin\alpha + S_2 \sin\alpha = \frac{G_1}{g} \cdot \frac{c^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The tangent to the curve is ┐ to the paper and Σ (tang. comps.) is evidently $= 0$, whence the tangential acceleration must be zero; i.e., c is constant, as we have assumed all along.

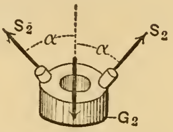


FIG. 85.

With G_2 free, in Fig. 85, we have *balanced forces;* whence Σ (vert. compons.) $= 0$; i.e.,

$$2S_2 \cos \alpha - G_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By elimination, noting that $c = 2\pi r u$,

$$\frac{c^2}{gr} = \frac{(G_1 + G_2)\tan\alpha}{G_1}; \text{ i.e., } \frac{r}{\tan\alpha}, = \overline{AC}, = \frac{g}{4\pi^2 u^2} \cdot \frac{G_1 + G_2}{G_1}.$$

We note, therefore, that the required distance AC is *independent of the length l* and is *inversely proportional to the square of the number of revolutions per unit time.* (A similar result was found with the simple conical pendulum; see p. 78, M. of E.)

Hence, numerically, with the foot, pound, and second (so that $u = \frac{120}{60} = 2$ revs. per sec.),

$$\overline{AB}, = 2\overline{AC}, = \frac{2 \times 32.2}{4 \times 9.87 \times 4} \cdot \frac{8 + 12}{8} = 1.018 \text{ ft.; or } 12.22 \text{ in.}$$

69. Example 17. Cannon-ball under Gravity and Air-resistance.—A round cannon-ball, weighing 24 lbs., is at a certain point of its path moving with a velocity of $v = 800$ ft. per second in a direction making an angle of 20° below the horizontal. The resistance offered to it by the air at this speed is 80 lbs. and acts in the line of motion, since the body is round and has no motion of rotation. Required the amount and position of the (ideal) resultant force R. See Fig. 86.

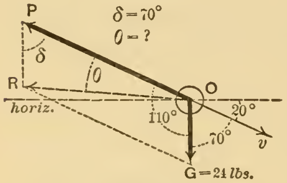


FIG. 86.

Since there are only two forces acting on the ball, P and G, R must have an amount and position determined by the diagonal of the parallelogram formed on P and G. See figure for the known angles. Hence (from formula on p. 7, M. of E.)

$$R = \sqrt{P^2 + G^2 + 2PG\cos 110°}, \quad \text{i.e.,}$$

$$R = \sqrt{80^2 + 24^2 + 2 \times 80 \times 24 \times (-0.3420)} = 75.25 \text{ lbs.}$$

To find the angle θ, note that in the triangle PRO we have $\sin\theta : \sin 70° :: G : R$; whence $\sin\theta = \frac{24}{75.25}(0.9397)$; i.e., $\theta = 17° \; 26'$, and hence R is $2° \; 34'$ above the horizontal.

Since the action-line of R is not coincident with the line of motion of the ball at this instant, *the path of the ball must be curved*, the radius of curvature at this point depending on the mass, on the square of the velocity, and on the value of the normal component of R; while the rate of retardation of the velocity (negative tangential acceleration) depends on the mass and the tangential component of R. (See next example.)

70. Example 18. Ball in Curved Path. Radius of Curvature, etc. (Fig. 87).—A large ball weighing 200 lbs. (so that its mass $= M = 200 \div 32.2 = 200 \times 0.031 = 6.2$, in the foot-pound-second system of units) at a certain point of its path has a velocity of 700 ft. per sec., the resultant force R at this instant being $= 300$ lbs. and making an angle of 140° with the direction (see v in figure) of motion.

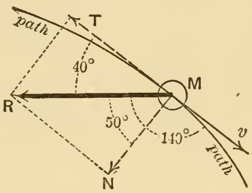


FIG. 87.

Required the radius of curvature, r, at this point of the path, and also the tangential acceleration.

By a rectangular parallelogram of forces we resolve R along the tangent and normal, obtaining for its tang. compon.,

$$T, = R \cos 140°, = 300 \times (-0.76604) = -229.812 \text{ lbs.},$$

while $N, = R \sin 140°, = 300 \times 0.6428 = 192.84$ lbs.

From eq. (5), p. 76, M. of E., Σ (norm. comps.) $= Mv^2 \div r$, and Σ (tang. comps.) $= Mp_t$, whence

$$r = \frac{6.2 \times (700)^2}{192.84} = 15{,}750 \text{ ft.};$$

and $$p_t = \frac{-229.812}{6.2} = -37.06 \text{ ft. per sec. per sec.}$$

This last value means that the retarding effect of the component T is such that *if* the rate of retardation remained constant for one second, at the close of that second the velocity in the path would be 700 — 37.06 = 662.94 ft. per sec.

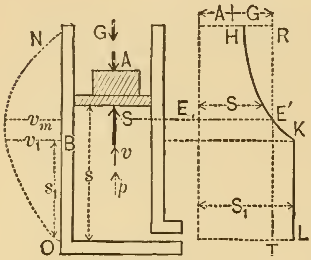


FIG. 87*a*.

71. Example 19. Steam Working Expansively and Raising a Weight.—In Prob. 4 of p. 61, M. of E., supposing the boiler-gauge to read 80 lbs. per sq. in. (above one atmosphere) and the total length of stroke, $s_n = ON$, to be 16 inches, with cut-off at one third stroke (so that $s_1 = \frac{1}{3}$ of 16 in.), the diameter of piston being 10 inches; how great a weight G can be raised if the (circular) piston is to come to rest at the end of the stroke, having started from rest at the beginning of the stroke? Required also the time occupied from O to B, and the position of the piston when its velocity is a maximum.

From p. 62 we have the equation

$$S_1 s_1 \left[1 + \log_\epsilon\left(\frac{s_n}{s_1}\right)\right] = As_n + Gs_n, \quad . \quad . \quad . \quad (2)$$

now to be solved for G. $s_n = \frac{4}{3}$ ft.; $s_1 = \frac{4}{9}$ ft., and hence the ratio $s_n : s_1 = 3$. The area of piston $= \pi r^2 = \frac{22}{7} \times (5)^2 = 78.57$ sq. in.

$\therefore$ Air-pressure above piston, $= A$, = constant = 78.57 × (15 lbs. per sq. in.) = 1178 lbs.; while the steam-pressure under piston while it is passing from O to B, $= S_1$, = 78.57 × 95 = 7464.15 lbs. Noting that $\log_\epsilon$ = common log × 2.302, we have from eq. (2) (using the foot, pound, and second)

$$7464 \times \tfrac{4}{9}[1 + 2.302 \times .47712] = 1178 \times \tfrac{4}{3} + G \times \tfrac{4}{3}.$$

Solving, $G = 4044$ lbs. (so that $M = 4044 \times .031 = 125.36$).

The acceleration from O to B is constant and =

$p_1 = (S_1 - A - G) \div M = 2241 \div 125.36 = 17.88$ ft. per sec. per sec.; and [eq. (2), p. 54, M. of E.] $s_1 = \frac{1}{2}p_1 t_1^2$; hence

$$\textit{time from } O \textit{ to } B = t_1 = \sqrt{\frac{2 \times \frac{4}{9}}{17.88}} = \sqrt{.0496} = 0.222 \text{ sec.}$$

Above B, the steam-pressure S diminishes, and when at some point m it has become $= A + G$, i.e., to 5222 lbs., the resultant or accelerating force, $S - (A + G)$, is zero; above this point m that force is negative, i.e., the velocity diminishes, and hence the velocity is a maximum at m. Let s_m be the distance of m from O; then from Boyle's Law $s_1 : s_m :: 5222$ lbs. $: S_1$, whence

$$s_m = \tfrac{4}{9}\cdot\tfrac{7464}{5222} = 0.635 \text{ feet}, = 7.620 \text{ in.}$$

72. Example 20. Ball Falling on Spring (Fig. 88).—A ball weighing two pounds (G) falls freely from rest, and after falling 5 ft. ($= h$) comes in contact with the head of a spring, which it gradually compresses during its further descent until brought to rest again momentarily (at m_1). The resistance (P) of the spring is proportional to the depth of compression (s) and is 60 lbs. (P_0) at the end of the first inch (s_0). (Provision is made against side-buckling.) Required the maximum compression $\overline{m_0 m_1}$, $= s_1$.

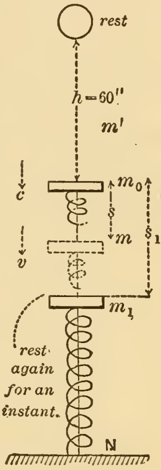


FIG. 88.

At the end of the free fall the velocity of the balls is $c = \sqrt{2gh}$ (i.e., $c^2 = 2gh$), since so far there is but one force acting, its own weight G. At any distance s, however, below m_0 (the point of first contact) the resultant downward force is $G - P$, P being the upward pressure of the head of the spring against the ball at this instant, and the acceleration is therefore variable and is $p =$ force $\div$ mass, $= (G - P) \div (G \div g)$. Let down be positive. Substituting in $vdv = pds$, noting that $P : P_0 :: s : s_0$, and then integrating between the points m_0 and m_1, we have

$$\frac{1}{g}vdv = ds - \frac{1}{G}Pds; \quad \text{and} \quad \frac{1}{g}\int_c^0 vdv = \int_0^{s_1} ds - \frac{P_0}{Gs_0}\int_0^{s_1} sds.$$

$$\frac{1}{g}\left[0 - \frac{c^2}{2}\right] = (s_1 - 0) - \frac{P_0}{Gs_0}\left[\frac{s_1^2}{2} - 0\right]; \text{ or, } -h = s_1 - \frac{P_0}{Gs_0}\cdot\frac{s_1^2}{2}.$$

Numerically, with the *inch*, *pound*, and *second*,

$$-60 = s_1 - \frac{60}{2 \times 1}\cdot\frac{s_1^2}{2}; \quad \text{whence} \quad s_1^2 - \frac{1}{15}s_1 = 4.$$

Finally, $\overline{m_0 m_1}$, $= s_1$, $= +2.03$ inches (and -1.96 in.).

The negative result refers to a point (call it m') 1.96 in. above m_0. This is the position where the ball would momentarily come to rest for the second time, if it adhered to the head of the spring after the latter had regained its natural length, supposing the lower end of the spring to be fixed. This motion of the ball while in contact with the spring is really *harmonic*, whose central point, from which the "displacement" would be reckoned, is $\frac{1}{30}$ of an inch below m_0, i.e., at the point where the pressure of the spring $=$ 2 lbs. (the weight of ball), so that as the ball passes that point its acceleration is zero and the velocity a maximum. This point is midway between m_1 and m'.

72a. The Engineer's "Mass."—The engineer measures the mass of a body (in case a problem connected with its *motion* is under treatment) by the fraction, *weight ÷ accel. of gravity*; or $G \div g$. This is not scientific, but is so firmly rooted in engineering practice that no different measure can well supplant it. It seems to imply that the amount of matter in a body depends on the existence of the attraction of gravitation; whereas, of course, such is not the case. This measure arises from the fact that a convenient way for the engineer to determine the magnitude of any force P (or resultant) acting on a body and producing an acceleration (p) of its velocity is to compare it with the force of gravity exerted on the body, whether the circumstances of the problem are affected by gravitation or not. In the phrase *force = mass × accel.*, or $P = Mp$, the word mass is simply a *name* given to the fraction $G \div g$. the origin of which is as follows:

In the actual problem the force P produces an acceleration $= p$ in the velocity of the body. In the *ideal* experiment of allowing the same body to have a free fall *in vacuo* we know that the only force *would be* the weight G, and that the resulting acceleration *would be* g; and since the forces must be proportional to the accelerations, we have (§ 54, M. of E.)

$$P : G :: \textit{actual } p : \textit{ideal } g; \quad \text{or,} \quad P = \frac{G}{g} p.$$

In other words, the engineer uses the gravitation measure of a force (p. 48, M. of E.).

CHAPTER V.

MOMENT OF INERTIA OF PLANE FIGURES.

72b. Phraseology.—Unless otherwise specified, we are to understand by "moment of inertia of a plane figure" the *rectangular* moment of inertia; i.e., the axis of reference lies in the plane of the figure (and not ┐ to it as with the "polar" moment of inertia). This is a useful function of the plane figure, to be used in the theory of beams under bending strain.

73. Moment of Inertia of Section of I-beam (Corners not Rounded).—Fig. 89 shows the form and dimensions of the section, which is symmetrical about each of the axes X and Y, and is for present purposes subdivided into three rectangles and four right triangles. Making use, then, of results obtained for those elementary forms, and of the transferral formula between the gravity axis of any figure and a parallel axis (see p. 94 and eq. (4), p. 93, of M. of E.), we have for the moment of inertia about axis X

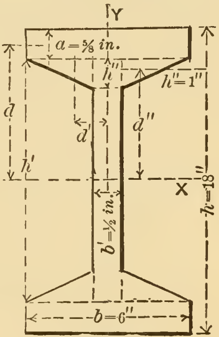


FIG. 89.

$$2\left(\frac{ba^3}{12} + ba\,.\,d^2\right) + \frac{b'h'^3}{12} + 4\left[\frac{(\frac{1}{2}b - \frac{1}{2}b')h''^3}{36} + \frac{b - b'}{2} \cdot \frac{h''}{2} \cdot d''^2\right].$$

That is, numerically,

$$2\left[\tfrac{1}{12}\cdot(6)\cdot(\tfrac{5}{8})^3 + 6\cdot\tfrac{5}{8}\cdot(8\tfrac{11}{16})^2\right] + \tfrac{1}{12}\cdot\tfrac{1}{2}\cdot(16\tfrac{3}{4})^3 + 4\left[\tfrac{1}{36}\cdot\tfrac{11}{4}\cdot(1)^3 + \tfrac{11}{8}\cdot(8.04)^2\right]$$

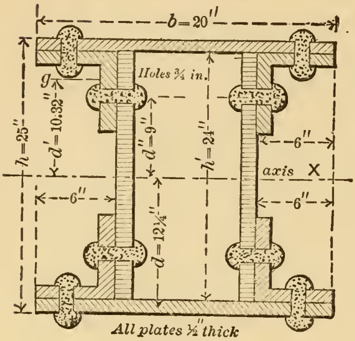
$$= 566 + 195.8 + 356 = 1117.8 \text{ bi. in.} = I_x.$$

Similarly, the moment of inertia about the axis Y is

$$2\cdot\frac{ab^3}{12} + \frac{h'b'^3}{12} + 4\left[\frac{1}{36}h''\left(\frac{b}{2} - \frac{b'}{2}\right)^3 + \frac{h''}{2}\cdot\frac{(b - b')}{2}\cdot d'^2\right]$$

$$= 2\cdot\tfrac{1}{12}\cdot\tfrac{5}{8}(6)^3 + (16\tfrac{3}{4})\tfrac{1}{12}(\tfrac{1}{2})^3 + 4\left[\tfrac{1}{36}(1)(\tfrac{11}{4})^3 + \tfrac{1}{2}\cdot\tfrac{11}{4}(1)(\tfrac{7}{6})^2\right]$$

$$= 22.5 + 0.17 + 9.79 = 32.46 \text{ bi-quad. in.} = I_y.$$



All plates ½" thick

FIG. 90.

74. Moment of Inertia of a Section of a Built Box-beam (Fig. 90). —The beam is composed of two "*flange-plates*" (upper and lower), two vertical "*stem-plates*," and four "*angle-bars*" of equal legs, riveted together. See figure for notation and dimensions. Required the moment of inertia of the whole section about X, its horizontal gravity axis of symmetry.

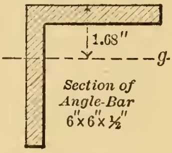


FIG. 91.

In Fig. 91 we have the location of the gravity axis g (parallel to X) of a single "angle" section, according to the hand-book of the New Jersey Steel and Iron Co., so that from the distance 1.68 in. we compute the $d' = 10.32$ in. of Fig. 90, or distance of axis g from axis X. From the same book we find that the I_g of the angle-section is 20 bi. in. (very nearly), and its area $F' = 5.75$ sq. in. Let $t =$ thickness of all plates $= \frac{1}{2}$ in.; and $t' =$ diam. of rivet-holes $= \frac{3}{4}$ in.

First, neglecting the rectangular gaps made by the rivet-holes, we have the I_x's of the various component sections as follows:

$$\text{(four "angles")} \ldots 4[I_g + F'd'^2)] = 4[20 + 5\tfrac{3}{4}(10.32)^2] \qquad = 2528;$$

$$\text{(flange-plates)} \ldots\ldots 2[\tfrac{1}{12}bt^3 + btd^2)] = 2[\tfrac{20}{12}\cdot(\tfrac{1}{2})^3 + \tfrac{20}{2}(12\tfrac{1}{4})^2] = 3001;$$

$$\text{(stem-plates)} \ldots\ldots 2\left[\frac{th'^3}{12}\right] \qquad = 2\left[\frac{\frac{1}{2}(24)^3}{12}\right] \qquad = 1152;$$

making a total of 6681 bi. in.

Treating the small rectangles left by the rivet-holes as concentrated in their respective centres of gravity [and thus neglecting their local (gravity) moments of inertia],

$$\left.\begin{array}{c}\text{Subtractive}\\ I_x \text{ due to}\\ \text{rivet-holes}\end{array}\right\} = 4[(2t)\cdot t'd''^2] + 4\left[(2t')t\left(\frac{h'}{2}\right)^2\right] = 243 + 432 = 675.$$

Hence, I_x of actual section $= 6681 - 675 = 6006$ bi. in.

It will be noticed that the first term, $\frac{1}{12}bt^3$ [or local (gravity) moment of inertia], in the I_x of the section of a *flange-plate*, above, is very small compared with the second, or "transferral term" $(bt \,.\, d^2)$. This is due to the fact that all parts of a thin flange-plate section are very nearly at the *same distance* from the final axis of reference, X. In such a case it is customary to neglect the local term, as no practical error results from so doing.

75. Moment of Inertia of Irregular Curve-bounded Plane Figures by Simpson's Rule (see § 93, M. of E.).—If an exact result for the I_x of the figure shown in Fig. 92 were desired, we might first conceive of its subdivision into narrow strips parallel to X, of variable length v and *infinitesimal* width dz, then express its I_x as $\int$(small area) $\times\, z^2$, or $\int (vdz)z^2$, $= \int (z^2v)dz$; and finally perform the integration, *if v were an algebraic function of z*. If such is not the case, however, (or if such is the case but the integration not practicable,) we can resort to Simpson's Rule (§ 15), noting that the u, x, and dx of that rule correspond to the (z^2v), z, and dz, respectively, of the present problem. We divide the whole height of the figure (from the axis X) into an *even* number n of *equal* parts, and through each point of division draw a parallel to X, thus determining a series of widths, v_0, v_1, v_2, etc. For example, this construction being made for the *upper* part of the rail-section in Fig. 93, with $n = 6$, we have its I_x, $= \int (z^2v)dz$, approximately $=$

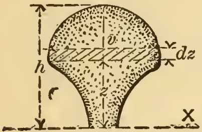


FIG. 92.

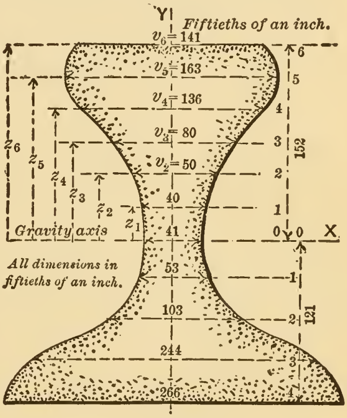


FIG. 93.

$$\frac{z_6 - 0}{3 \times 6}[z_0{}^2v_0 + 4(z_1{}^2v_1 + z_3{}^2v_3 + z_5{}^2v_5) + 2(z_2{}^2v_2 + z_4{}^2v_4) + z_6{}^2v_6].$$

With numerical substitution, therefore, the lengths marked in the figure having been scaled in fiftieths of an inch (noting that

$z_0 = 0$, $z_1 = \frac{1}{6}z_6$, $z_2 = \frac{2}{6}z_6$, $z_3 = \frac{3}{6}z_6$, etc.), we have, as the I_x of the portion of the rail-section lying above the axis X,

$$\frac{(152)^3}{3 \times 6^3}[0 + 4(1^2 \times 40 + 3^2 \times 80 + 5^2 \times 163) + 2(2^2 \times 50 + 4^2 \times 136) + 6^2 \times 141]$$

bi-quad. fiftieths; which divided by $(50)^4$, or 6,250,000, gives

$$I_x \text{ for upper part} = 25.27 \text{ bi. in.}$$

Similarly, the vertical height of the lower part being divided into four equal lengths, we have for the I_x of that part (nearly)

$$\frac{(121)^3}{3 \times 4^3}[0 + 4(1^2 \times 53 + 3^2 \times 244) + 2(2^2 \times 103) + 4^2 \times 266]$$

$= 129{,}870{,}000$ bi. fiftieths. Dividing by $(50)^4$, we have 20.77 bi. in.; so that the total I_x of the complete rail-section $= 20.77 + 25.27 = 46.04$ bi. in.

X is a gravity axis parallel to the base of the rail-section, and has been located by cutting out the shape from card-board and balancing on a needle-point.

If the plane figure is of such a form that a division into strips perpendicular to, and *all* terminating in, the axis of reference (X) is convenient, the exact calculus form for its I_x is $\frac{1}{3}\int y^3 dx$ (see latter part of § 93, M. of E.), for each strip is an elementary rectangle. See Fig. 94.

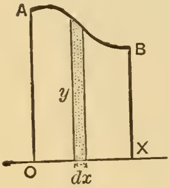


FIG. 94.

If Simpson's Rule is to be applied, divide the base OX of the figure into an even number, n, of equal parts and scale the extreme ordinates $y_0 = \overline{AO}$ and $y_n = \overline{BX}$, also the intermediate ordinates y_1, y_2, etc., at the points of division. With $n = 6$, for example, we have as an approximation

$$I_x = \frac{\overline{OX}}{9 \times 6}[y_0^3 + 4(y_1^3 + y_3^3 + y_5^3) + 2(y_2^3 + y_4^3) + y_6^3].$$

76. Graphical Method* for the Gravity Axis and Moment of Inertia of a Plane Figure (Fig. 95).—$A''B''C''$ is the figure in question (drawn in full size). It is required to construct the special gravity axis, R, that is parallel to the base $A''B''$, and to obtain the moment of inertia about that axis.

Divide the figure into strips parallel to $A''B''$, of small width

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* From Ott's Graphical Statics.

(no width being more than one eighth, say, of the total width ┐ to $A''B''$). These widths need not be equal.

Through the centres of gravity of the strips (1, 2, 3, etc.) draw indefinite lines ($1\,..\,1'$, $2\,..\,2'$, etc.) || to $A''B''$ (with most strips it is accurate enough to take the centre of gravity midway between the sides). Along any right line $O'I$, parallel to $A''B''$, lay off the lengths $O'A$, AB, BC, etc., proportional, respectively, to the areas F_1, F_2, etc., of the successive strips, in the order and position shown (in most instances* each such area may be assumed proportional to the length of the strip measured through the centre of gravity). Through O' and I draw lines at 45° with $O'I$, as shown, to determine the "pole" O, from which the "rays" OA, OB, etc., are now drawn. Then through any convenient point Z draw ZC''' || to OO'. From the intersection a with $1\,..\,1'$ draw ab || to OA to intersect $2\,..\,2'$ in some point b, then bc || to OB, and so on; until finally, through i, iC is drawn || to IO to determine C by intersecting ZC'''. The required gravity axis R passes through C || to $A''B''$. (For proof consult § 376, M. of E.)

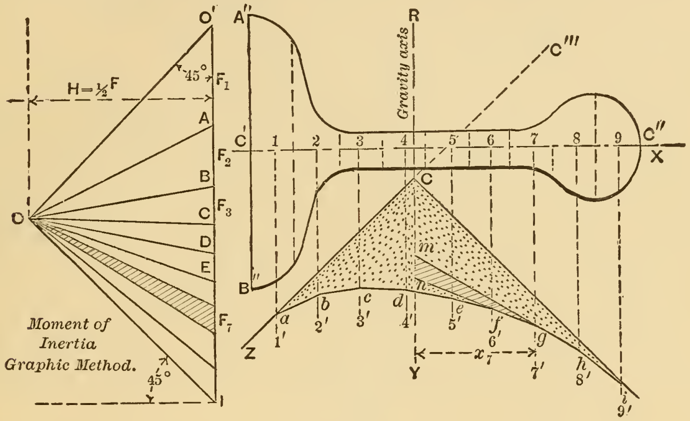

FIG. 95.

The *moment of inertia about axis R* may be obtained by *multiplying together the area of the given figure $A''B''C''$* (call it F) *by the area* (call it F') *of the "inertia-figure"* (i.e., the area included between the two lines Ca and Ci, and the broken line, or "equilibrium-polygon," *abcdefghi*; shaded in Fig. 95). The proof of this relation (strictly true only for infinitely narrow strips) is as follows:

---

* When the strips are narrow and of *equal* width.

At any vertex of the equilibrium-polygon, as at g, there are two segments meeting; prolong them to intersect the gravity axis R in some two points, as m and n. Then mng is a triangle with base $m\,..\,n$ (call it k_7) and altitude x_7. But on the left of Fig. 95 the shaded triangle OF_7 is evidently similar to mng; whence the proportion $k_7 : x_7 :: F_7 : \frac{1}{2}F$; i.e., $F_7x_7 = \frac{1}{2}Fk_7$. Multiplying by x_7, we have $F_7x_7^2 = F(\frac{1}{2}x_7k_7)$.

Now $F_7x_7^2$ is the moment of inertia (about R) of strip No. 7 (considered infinitely narrow), and $\frac{1}{2}x_7k_7$ is the area of the triangle mng. We have therefore proved that the moment of inertia of any one strip is equal to the product of the *whole* area F of the given plane figure by the area of a triangle like mng and obtained in a similar manner. If all the triangles like mng were drawn, their united areas would evidently be that of the "*inertia-figure*" *abcdefghi-C-a*. Hence the sum of the moments of inertia of all the strips, i.e., the moment of inertia of the whole figure $A''B''C''$, is

$$I_R = \left\{ \begin{matrix} \textit{area of plane} \\ \textit{figure } A''B''C'' \end{matrix} \right\} \times \left\{ \begin{matrix} \textit{area of the} \\ \text{“}\textit{inertia-figure}\text{”} \end{matrix} \right\}.$$

In practice these areas are most conveniently and accurately obtained by means of a planimeter; otherwise, subdivision into small trapezoidal strips may be resorted to. If the scale of the drawing is one half of the actual size the result must be multiplied by 16, i.e., the *fourth power* of 2; and similarly for other ratios.

By the use of a planimeter with this actual figure (Fig. 95) the results $F = .90$ sq. in., and $F' = .42$ sq. in., have been obtained; therefore $I_R = 0.378$ bi. in.

CHAPTER VI.

DYNAMICS OF A RIGID BODY.

77. Example of Rotary Motion. Axis Fixed and Horizontal (Fig. 96).—The body AB consists of an irregular solid and a light drum, rigidly connected and mounted on a horizontal axle whose journals are 2 inches $(= 2r)$ in diameter, at C; the radius of the drum being $a = 5$ in. The weight of AB is 150 lbs. $= G_1$, and its centre of gravity is situated in the axis of rotation (hence G_1 has no effect as regards changing rotary velocity, having always a zero moment about the axis). The weight G, $= 20$ lbs., is attached to an inextensible cord which is wound on the drum and whose weight is neglected. A constant friction F, $= 10.5$ lbs., acts at the circumference of the journals.

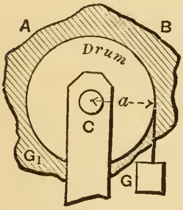

FIG. 96.

It is required to compute the radius of gyration k_c of the rotating body by the experiment of noting the time t_1 occupied by G in sinking a measured distance s_1, from rest. Suppose $t_1 = 5$ sec. and $s_1 = 10$ ft.

At any instant during this motion, the tension in the cord being S, we have for the angular acceleration θ of AB, which is shown free in Fig. 97, taking moments about axis C,

$$\theta\cdot\frac{G_1}{g}k_c^2 = Sa - Fr, \quad . \quad . \quad . \quad (1)$$

while at the same instant the downward linear acceleration p, $= \theta a$, of G, enters into the relation involving the sum of downward forces for G as a free body, viz.:

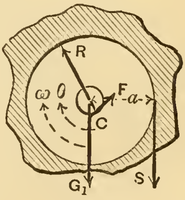

FIG. 97.

$$G - S = mass. \times accel. = \frac{G}{g}p. \quad . \quad . \quad . \quad . \quad (2)$$

Solving for p, we would find it constant, as also S; hence G has a uniformly accelerated rectilinear motion, and AB a uniformly accelerated angular motion. Therefore (with foot, pound, and second), since (from eq. (2), p. 54, M. of E.) $s_1 = \frac{1}{2}pt^2$, and hence $p = 2s_1 \div t_1^2$, we have $p = 20 \div 25 = 0.80$ and is the constant linear acceleration, not only of G, but for any point in the surface of the drum, so that the angular acceleration of body AB is $\theta = p \div a = 0.80 \div \frac{5}{12} = 1.92$ radians per sec. per sec. From eq. (2) we now have $S = 20 - (20 \times .80) \div 32.2 = 19.504$ lbs. as the constant tension in the cord; and finally, from eq. (1),

$$k_c^2 = \frac{[19.504 \times \frac{5}{12} - 10.5 \times \frac{1}{12}]32.2}{1.92 \times 150} = 0.812 \text{ sq. ft.};$$

i.e., the rad. of gyration $k_c = 0.901$ ft. = about 10.8 inches.

78. Solution of Example of Compound Pendulum. (For the statement of the example see p. 121, M. of E.) Fig. 98. ($h = 6''$, $r = 1.2''$. Locate axis oo so that the time of oscillation t' shall be $\frac{1}{2}$ sec.) Referred to the horizontal-gravity axis cc we have for the solid cone (from § 101, M. of E.) $I_c = \frac{3}{20}M[r^2 + \frac{1}{4}h^2]$, i.e., $k_c^2 = \frac{3}{20}[r^2 + \frac{1}{4}h^2] = \frac{3}{20}[1.44 + \frac{36}{4}] = 1.566$ sq. in. With the *inch* and *second* $g = 386.4$, and $gt'^2 \div 2\pi^2 = 4.889$. Hence, from eq. (1), p. 120, M. of. E.,

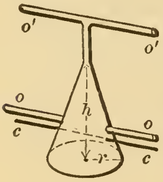

FIG. 98.

$$s = 4.889 \pm \sqrt{23.90 - 1.556} = +9.615, \quad \text{and} \quad +0.163 \text{ in.}$$

That is, with either oo or $o'o'$ as axis of suspension, co being made $= 0.163$ in., and $co' = 9.615$ in., the duration of an oscillation (of small amplitude) will be $\frac{1}{2}$ sec. of time. As a check we note that $co + co' = 9.778$ in., which $= 2 \times 4.889$ = length of a simple pendulum beating half-seconds (see above and also eq. (3), p. 119, M. of E.).

79. Points of Maximum and Minimum Angular Velocities in Motion of Crank-pin of a Steam-engine.—Fig. 99 shows the results of applying the tentative graphic method mentioned in the middle of p. 124, M. of E. In making trials for successive positions of the connecting-rod, advantage can be taken of the fact that if the point be found where the axis of the connecting-rod (prolonged if necessary) intersects the vertical line (vertical in this instance), drawn through the centre C of the crank circle, the distance of that point from C represents the amount of the tangential component, T, of the force P', on the same scale on which the length, $C\,..\,n$, of the crank would represent the force P. Hence various positions of the connecting-rod are drawn until those two are found for which the distance mentioned above is $\frac{35}{55}$, i.e., $\frac{7}{11}$, of radius $C\,..\,n$.

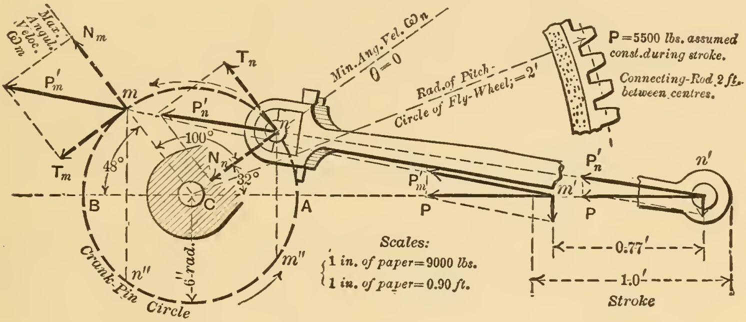
Max. Angul. Veloc. ω m
N m
P′ m
m
T m
T n
P′ n
Min. Ang. Vel. ω n
θ = 0
Rad. of Pitch-Circle of Fly-Wheel; = 2′
P = 5500 lbs. assumed const. during stroke.
Connecting-Rod 2 ft. between centres.
100°
N n
48°
32°
n
B
C
A
P′ m
P
m′
P′ n
n′
P
Scales:
1 in. of paper = 9000 lbs.
1 in. of paper = 0.90 ft.
0.77′
1.0′
Stroke
m″
n″
6 rad.
Crank-Pin Circle

FIG. 99.

In the working of most engines the steam is used *expansively* so that P is variable, being greatest near the beginning of the stroke. This would cause the points n and m to be nearer to A; and similarly, on the return-stroke, n'' and m'' nearer to B.

80. Rapid Rotation of a Body on a Fixed Axis. Effect on Bearings.—So long as a wheel or pulley in a machine is perfectly symmetrical about the axis of rotation the pressures on the bearings are due simply to the weight of the pulley and the pulls or thrusts of cogs, belts, cams, etc., which may be acting on the pulley or on the shaft on which it is mounted; whether rotation is proceeding or not. But if, through any imperfection in the adjustment or mounting, the centre of gravity lies outside of the axis of rotation (its distance from that axis being $\bar{\rho}$), other pressures are brought upon the bearings, the same in amount *as if* the pulley were at rest and a force $= \omega^2 M\bar{\rho}$ acted through the centre of gravity, away from, and ┐ to, the axis of rotation.

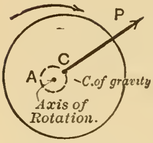

FIG. 100.

For example, let the pulley in Fig. 100, of weight = 644 lbs., be out of centre by one fourth of an inch and be rotating uniformly at the rate of 210 revolutions per minute, i.e., $\frac{7}{2}$ rev. per second. Its angular velocity is therefore $\omega = 22$ radians per second and the pressures on the bearings due to the eccentricity of the pulley at this speed is the same *as if* the pulley were at rest and a "*centrifugal force*" P, $= [484(644 \div 32.2)\cdot\frac{1}{48}]$, $= 201.6$ lbs., acted on the body, as shown in figure, where C is the centre of gravity. The pressures on the two bearings due to this ideal "centrifugal force" will be inversely proportional to their distances from the pulley-centre and parallel to P (aside from the pressures due to the weight of the pulley, action of belts, cogs, etc.).

The continual change of direction of these centrifugal pressures, as the wheel revolves, is likely to cause injurious vibrations in the framework of the machine.

81. "Centrifugal Couple." (Continuing the discussion of the last paragraph.)—Even if the axis of rotation *does* contain the centre of gravity (or rather *centre of mass* in this connection) of the revolving body; if, even then, we find that any plane containing the axis divides the body into two parts,* the right line connecting whose centres of mass is *not perpendicular to the axis*, the ideal centrifugal forces, $\omega^2 M_1\bar{\rho}_1$ and $\omega^2 M_2\bar{\rho}_2$, of these two component masses form a *couple*, causing pressures at the two bearings; which two pressures, themselves, form a couple. For example, Fig. 101, where the axis of the shaft AB and the centres of the two masses are in the same plane, and where the products $M_1\bar{\rho}_1$ and $M_2\bar{\rho}_2$ are equal, the centre of mass of the whole rigid body must lie in the axis of rotation (i.e., neglecting the mass of the shaft); but the line joining the centres of the component masses is not perpendicular to the axis AB. A and B are fixed bearings and a uniform rotation with angular velocity of ω is proceeding. Evidently the

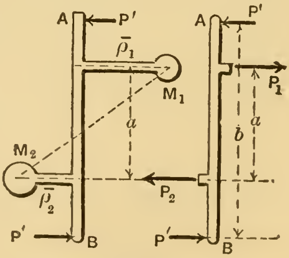

FIG. 101.

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* We here suppose each of these parts to be homogeneous and to be symmetrical about a plane passed through its center of gravity and perpendicular to the axis of rotation. If such is not the case, we must resort to the principles of §§ 122–122*c* inclus. of M. of E., to determine the pressures at the bearings.

action on the bearings is the same as if the body AB were at rest and were acted on by the two ideal centrifugal forces $P_1 = \omega^2 M_1 \bar{\rho}_1$ and $P_2 = \omega^2 M_2 \bar{\rho}_2$, in the positions shown in the figure (on the right). These forces are equal (since $M_1 \bar{\rho}_1 = M_2 \bar{\rho}_2$), forming therefore a couple, so that the reactions at the two bearings (pressures of bearings on shaft) would form a couple of moment $P'b = P_1 a$.

For example, let M_1 weigh 120 lbs.; M_2, 100; the distances $\bar{\rho}_1$ and $\bar{\rho}_2$ being 2 and 2.4 ft., respectively; while the angular velocity is 22 radians per second (210 revs. per min.). Also let $a = 3$ ft. and $b = 4$ ft.; then

$$P' = \frac{a}{b} P_2 = \frac{3}{4} \cdot (22)^2 \frac{100}{32.2} \times 2.4 = 2710 \text{ lbs.}$$

So long as the rotation is uniform these pressures P' lie in the plane of the axis and the two mass-centres; which plane, of course, is continually changing position.

82. Piles. (See p. 140, M. of E.)—As a practical matter it should be understood that as the head of the pile becomes broomed, or splintered, during the driving, the penetration occasioned by the blow may be much smaller than would otherwise be the case (only one quarter as much in some cases); hence it is economical of power that the head be adzed off occasionally, and especially should this be done just previously to the last few blows when the measurement of penetration is made, to be used in a formula for safe bearing load.

A formula for the safe load has been proposed (see p. 185, Eng. News, Feb., 1891) in which, besides adopting the divisor 6 of eq. (2), p. 140, M. of E., the value of s' is assumed one inch greater than that actually observed. In cases where the actual s' is small (as an inch or less) this allows a very wide margin of safety. If s' is large, the margin of safety is practically little more than that afforded by the divisor 6. This formula may be written:

$$\textit{Safe load} = \frac{1}{6} \cdot \frac{Gh}{s' + \textit{one inch}}; \quad \left\{ \begin{array}{l} \textit{or, for the} \\ \textit{lb. and ft.,} \end{array} \right\} = \frac{1}{6} \frac{Gh}{s' + \frac{1}{12}};$$

while if G is in lbs., h in feet, and s' in inches, it takes the form

$$\textit{safe load in lbs.} = \frac{2Gh}{s'+1}.$$

(See Baker's Masonry Construction, and Trautwine's Pocket Book, for many practical details in the matter of piles.)

In experiments in the laboratory of the College of Civil Engineering at Cornell University, with round oak stakes 2 in. in diameter and a 4-lb. hammer falling four feet, the actual bearing loads have been found to be from about one half to seven eighths of that given by the expression $Gh \div s'$.

83. Kinetic Energy of Rotary Motion. Numerical Example.—The grindstone in Fig. 102 has an initial angular velocity, ω_0, corresponding to 180 rev. per min., about its horizontal geometrical axis, which is fixed. How many turns will it make before it is brought to rest by the two frictions, at A and at C? The pressure of the plank OD on the stone at A is due to the weight of 180 lbs. suspended at rest at D. Assume that the friction, F', or tangential action of the plank against the stone, is always one third of the normal pressure, which is vertical and is evidently 360 lbs. (since the plank is a body at rest and therefore under a balanced system of forces, so that by moments about O we find the normal pressure to be double the 180 lbs.). Hence the friction F' is $=$ 120 lbs., and with regard to the plank is a force directed *toward the right;* but *toward the left*, as regards the revolving stone. The stone is a homogeneous right cylinder weighing 600 lbs., and of radius $= 2'$, its geometrical axis being the axis of rotation. Radius of journals at C is 1 in., and the mass of the projecting axle is neglected. The journal friction, F'', tangent to the circumference, is to be taken as constant and as being $\frac{1}{20}$ of the normal pressure, N'', on the journal.

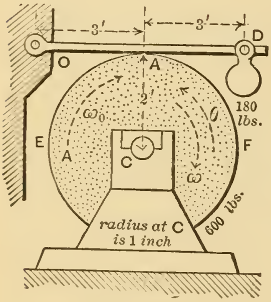

FIG. 102.

We now consider the stone free in Fig. 103. The forces acting are as shown, and the rotation clockwise with retarding angular velocity.

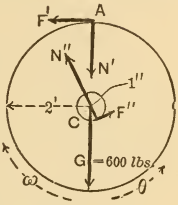

FIG. 103.

Though the system is not a balanced one as regards rotation, since the moment-sum about C is not zero but $= \theta M k_c^2$, it is such as regards motion of the whole mass vertically; for the centre of mass has a zero vertical acceleration, being at rest at all times. Hence we may put Σ (vert. comps.) $= 0$, and thus obtain, noting that the pressure N'' is practically vertical, $N'' = G + N'$; and hence $F'' = 960 \div 20 = 48$ lbs.

To apply here the principle of Work and Energy, as expressed in eq. (2) of p. 144, M. of E., the range of motion considered being that occupied by the stone in coming to rest, we note, Fig. 103, that G, N', and N'' are neutral; that F' and F'' (both assumed constant) are resistances; and that there are *no working forces.* Let u denote the unknown number of turns made in coming to rest, ds' an element of the path of the point A, and ds'' an element of the path of a point in circumference of the journal. With the *foot*, *pound*, and *second*, we have, therefore:

$$\left.\begin{matrix}\textit{Work}\\ \textit{done}\\ \textit{on } F'\end{matrix}\right\} = \int_0^n F'ds' = F' \int_0^n ds' = [120 \times u \times 2\pi \times 2] \text{ ft. lbs.};$$

$$\left.\begin{matrix}\textit{Work}\\ \textit{done}\\ \textit{on } F''\end{matrix}\right\} = \int_0^n F''ds'' = F'' \int_0^n ds'' = [48 \times u \times 2\pi \times \tfrac{1}{12}] \text{ ft. lbs.}$$

Since k_c^2 for the cylinder (see p. 99, M. of E.) is $= \frac{1}{2}r^2 = 2$ sq. ft., and the initial angular velocity is $\omega_0 = 2\pi(180 \div 60) = 18.85$ radians per second, we have

$$\left\{\begin{matrix}\textit{Initial Kinetic}\\ \textit{Energy of Stone}\end{matrix}\right\} = \tfrac{1}{2}\omega_0^2 M k_c^2 = \left[\tfrac{1}{2}(18.85)^2 \cdot \frac{600}{32.2} \times 2\right] \text{ ft. lbs.}$$

The final kinetic energy is to be zero and the work of the working forces is zero; hence from eq. (2), p. 144, M. of E.,

$$0 = 480\pi u + 8\pi u + [0 - (18.85)^2(18.6)]. \quad . \quad . \quad (3)$$

Solving this, we obtain $u = 4.33$ turns, for the stone to be brought to rest. Eq. (3) might be read: *the initial K. E. is entirely absorbed in the work of overcoming friction.*

(How would the *time* of coming to rest be determined?)

84. Work and Kinetic Energy. Motion of Rigid Body Parallel to a Plane. Numerical Example.—Conceive of a rigid body, or hoop, Fig. 104, consisting of two material points, M' and M'', so connected with a rigid, but imponderable, framework that the centre of gravity of the two material points lies at the centre of the circle formed by the outer edge of the frame. This outer circle, or wire, is to roll *without slipping* from a state of rest (centre at o) to a position 6 ft. lower, vertically (centre at n); q, in the figure, representing any intermediate position. The two masses are $M' = 2$ and $M'' = 3$ in the ft.-lb.-sec. system of units, their positions relatively to the frame being shown in the figure.

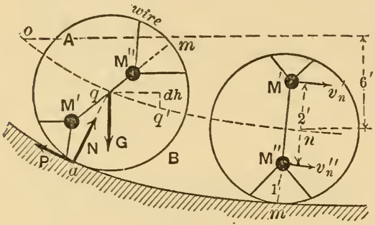

FIG. 104.

Assuming that in passing position n the two masses are in line with the point of rolling contact m (M' uppermost), required their respective linear velocities, v_n' and v_n'', at that instant. On account of the perfect rolling, each mass is at this final instant moving in a line ┐ to the line connecting it with m, and v_n' and v_n'' are proportional to these distances;

$$\text{i.e.,}\quad v_n' : v_n'' :: 3 \text{ ft.} : 1 \text{ ft.}; \text{ or } v_n' = 3v_n''.$$

Throughout this range of motion (o to n) the acting forces are G, = 161 lbs., = the combined weight of the material points applied at the centre of the hoop (see AB in Fig. 104) and the normal and tangential components, N and P, of the pressure of the fixed curved track or guide against the edge of the hoop at a. When any point of the hoop is in contact with the track, that point becomes the point of application of both N and P, and is *at rest*, being the one point of the hoop about which all others are turning for the instant ("centre of instantaneous rotation"), hence both N and P are *neutral forces.* As the centre of hoop passes from q to q' through an element of its path, G does the work Gdh; hence the total work done by G is $G\int dh =$

161 lbs. × 6 ft. = 966 ft.-lbs. of work. The initial K. E. of the whole body is = zero, and its final K. E., i.e., its $(\text{K. E.})_n$, is

$$\tfrac{1}{2}M'v_n'^2 + \tfrac{1}{2}M''v_n''^2 = \tfrac{1}{2}v_n''^2[9M' + M''].$$

$$\therefore\ 966 = \tfrac{1}{2}v_n''^2[18 + 3] - 0; \quad \text{or} \quad v_n''^2 = 92;$$

whence $v_n'' = 9.59$ ft. per sec.; and also $v_n' = 28.77$ ft. per sec.

[If a single material point were to slide from rest down a smooth fixed guide through this vertical height of 6 ft., its final velocity would be $(\sqrt{2 \times 32.2 \times 6} =)$ 19.65 ft. per sec.].

If the hoop in the above example were to slip on any part of the guide, instead of rolling perfectly, the force P would no longer be neutral; and if the edge of the hoop were to indent the surface without immediate and perfect recovery of form on the part of the latter, N and P would not be neutral. (See § 172, M. of E.)

85. Numerical Example. Work and Energy. Collection of Rigid Bodies. [Read the ten lines under eq. (XVI), p. 149, M. of E.]—Fig. 105 shows a body mounted on a fixed horizontal axis containing its centre of gravity, of weight = 48.3 lbs. (hence its *mass* = 1.5, using ft., lb., and sec.), its radius of gyration about the axis being $k = 3$ ft. This body includes a drum, as shown, from which a light inextensible cord may unwind until the point A of drum passes the position A' (the cord being firmly inserted), thus allowing the 10-lb. weight to sink vertically. C is a smooth fixed peg; there is no friction at the journals. The figure shows the initial position (rest). The 10-lb. weight begins to sink, and the velocities to accelerate, while the cord simply unwinds until A reaches the unwinding point D, beyond which the point of insertion will follow the arc DA', the portion of cord between it and C being straight.

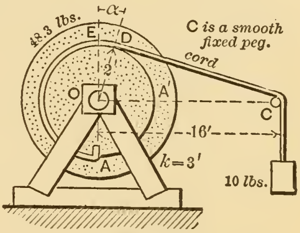

FIG. 105.

Required the (final) angular velocity ω_n of the rotating body when A reaches A'.

Considering free this collection of rigid bodies (two masses and cord), knowing that all *mutual* actions between them, if normal to surfaces, are neutral, and assuming no internal friction, we note that the only forces external to the collection are the gravity-force of 48.3 lbs. (neutral because its point of application does not move); the normal reactions of the bearings against the sides of the journals (these are neutral because the path of the point of application is always ┐ to the action-line of the force); the pressure of the peg against the side of the cord (neutral, for the same reason as that just mentioned); and the gravity-force of 10 lbs., which is the only working force. The path of the point of application of this last force is vertical and coincident with the action-line of the force, and the length of this path (which is equal to that of its own projection on the action-line) is equal to the total length of cord, s, that runs over the peg. Evidently $s = \pi \times 2$ ft. $= 6.28$ ft. (while A is passing to E), increased by the difference between the distance EC (considered straight; α is very small in this case) and the straight distance $A'C$;

$$\text{i.e., } s = 6.28 + \left[\sqrt{16^2 + 2^2} - 4\right] = 8.406 \text{ feet.}$$

As both the initial and the final Kinetic Energy of the 10-lb. mass are zero, and the initial K. E. of the rotating body is zero, the work $10 \times 8.406 = 84.06$ ft.-lbs., of the one working force, is entirely expended in generating the final K. E., $\frac{1}{2}\omega_n{}^2 Mk^2$, of the rotating body,

$\therefore 84.06 = \frac{1}{2}\omega_n{}^2 \times 1.5 \times (3)^2$; and $\omega_n = 3.53$ rad. per sec., which corresponds to $\omega_n \div 2\pi = 0.562$ rev. per sec.

As to the tension, S', induced in the chord as the 10-lb. weight reaches its lowest point and begins to ascend, which is when the point of insertion of the chord in the drum passes through the point A (in its circular path), we note that at that instant, since the chord cannot stretch, the horizontal acceleration of the insertion-point, which is its *normal* acceleration at A', must be the same* as the vertical acceleration of the 10-lb. weight, and this is upward, since the downward velocity is slackening. Hence the resultant force, $S' - G$, on G at this instant is upward, and $= (G \div g) \times$ (*normal accel. of a point moving with linear velocity* $= \omega_n \times 2$ ft. *in a curve whose rad. of curv. is* 2 ft.),

$$\text{i.e., } = (G \div g) \times (\omega_n \times 2 \text{ ft.})^2 \div 2 \text{ ft.} = 7.72 \text{ lbs.}$$

$$\therefore S' = 10 + 7.72 = 17.72 \text{ lbs.}$$

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* Approximate; the error is less the longer the distance $A'C$, compared with OA', strictly true only when the peg is at an infinite distance toward the right. (A somewhat similar approximation was made on p. 59, M. of E., in considering the connecting-rod of a steam-engine as infinitely long.)

At other parts of the motion the cord-tension is smaller. Let the student trace the remainder of the motion.

86. Work of Internal Friction.—The student should note well that the work spent on the friction between two rubbing parts of the collection of rigid bodies under consideration is formed by multiplying the value of the friction by the distance rubbed through by one of the parts on *the surface of the other*, independently of the actual path in space of any point of either body; for instance, in the examples of §§ 144 and 145, M. of E., the friction ($F'' = 400$ lbs.) between the crank-pin and its bearing (in the end of the connecting-rod), the range of motion of the mechanism being that corresponding to a half-turn of the crank, is multiplied by the length of a half-circumference of the crank-pin itself, viz., $\pi r''$, $= 0.314$ ft. (less than 4 inches), whereas the centre of the crank-pin describes a distance of $\pi r = 3.14$ ft. at the same time.

87. Locomotive at Uniform Speed.—Required the necessary total mean effective pressure P_0 in the cylinder of a locomotive on a level track to maintain constant the speed of a train whose resistance, R, at that speed is given. This resistance is due entirely to frictions of various kinds in the train and to the resistance of the air, and, the track being level, is equal to the tension in the draw-bar immediately behind the locomotive. Fig. 106 shows the collection of rigid parts forming the locomotive alone and the forces external to them, all mutual frictions being disregarded. For simplicity consider that there is only one cylinder and piston; and take, as the range of motion of the parts, that corresponding to one backward stroke of the piston, i.e., to a half-turn of the driving-wheels. The external forces are: P_0 on the piston; an equal P_0 on the front cylinder-head; Y, Y, the pressures of the rails against the truck-wheels; S, S, those of the track against the driving-wheels, which are supposed not to slip on the track; G, the weight of the locomotive; and R, the tension in the draw-

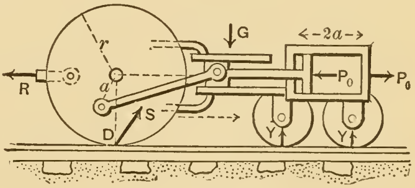

FIG. 106.

bar. Of these, the Y's and the S's are neutral for reasons given in the example of § 84 (perfect rolling); G also is neutral, the elements of its path being always ┐ to the action-line of the force; R is a resistance, overcome through a distance $= \pi r$; the P_0 on the front cylinder-head is a working force, its point of application describing a horizontal path of length $= \pi r$; while the P_0 on the piston is a resistance whose application-point moves forward in absolute space a distance $= \pi r - 2a$. The kinetic energy of the mechanism being the same at the end as at the beginning of the stroke, the K. E. terms cancel out from the equation of work and K. E., leaving

$$P_0 \pi r - P_0(\pi r - 2a) - R\pi r = 0; \quad \therefore\ P_0 = \frac{R\pi r}{2a},$$

or $P_0 =$ the draw-bar resistance multiplied by the half-circumference of the driver and divided by the length of one stroke. For example, if a 200-ton train offers a resistance of 10 lbs. per ton (*in the draw-bar*) at a speed of 20 miles per hour, then with 2.5 ft. radius for the drivers and 1 ft. radius for the crank-pin circle, we have $P_0 = 7857$ lbs.; i.e., with two cylinders, 3928 lbs. for each piston. Suppose each piston to have an area of 100 sq. in., then the average amount by which the steam-pressure on one side should exceed the atmospheric pressure on the other is 39.28 lbs. per sq. in. As steam is ordinarily used expansively when the train is under full headway, this means an initial steam-pressure at the beginning of stroke of perhaps 80 or 90 lbs. per sq. in. above the atmosphere (as a roughly approximate illustration). Above, we have assumed the drivers not to slip on the rails. Slipping will not occur ordinarily if R is less than about one fifth of the total weight resting on the drivers.

88. Locomotive Tractive Effort at Starting. Track Level.— When the train is once in motion the steam-pressure on the piston in the latter part of the stroke is much smaller than the average, so that the speed slackens temporarily, the great mass of the train acting as a fly-wheel. To start, however, this steam-pressure must reach a certain minimum amount which we call P. For instance, with one engine at its dead-centre, the duty of starting devolves on the other alone, which is then in the mid-stroke position (nearly), its crank being vertically over (or under) the driver-axle, and the horizontal component of the pull on the crank-pin is $= P$. Fig. 107 shows the driving-wheel as a free body, the vertical components of the acting forces being omitted, as having no moments about D, the point of contact with the track. (Steam is now pressing on the left face of the piston, not shown, and on the hinder (left end) end of cylinder-head (see Fig. 106 for direction of motion, etc.). The action of the driver is that of a lever whose fulcrum is at D (no slip). P' is the horizontal pressure of the locomotive-frame against the driver-axle, as induced by the pull P; and by moments about D we have $P'r = P(r + a)$; i.e., $P' = P + \frac{a}{r}P$. Considering now the locomotive-frame in Fig. 108, we find a forward force P' at A, the bearing of the driver-axle; the draw-bar resistance R, backward; and a backward steam-pressure $= P$ on the hinder cylinder-head at B; whence, for equilibrium (i.e., more strictly, for a *very small acceleration*), $P' - P = R$, and finally $P = \left(\frac{r}{a}\right)R$, as the necessary total effective steam-pressure on the one piston to start the train when the draw-bar resistance is R (neglecting the resistances in the locomotive itself). (No vertical forces shown.)

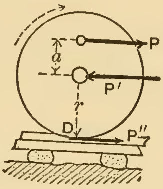

FIG. 107.

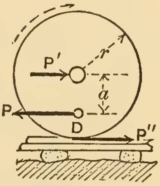

FIG. 109.

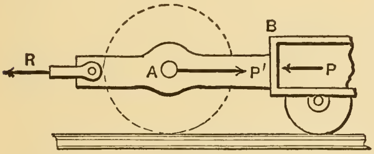

FIG. 108.

Similarly, if the crank-pin were vertically under the axle, as in Fig. 109, we obtain for the pressure at axle (induced by P) the value $P' = P - \left(\frac{a}{r}\right)P$, [since, from moments about D, $P'r = P(r - a)$]. Hence, as regards the locomotive-frame, we now find a *backward* force P' at the axle-bearing; but there is now a *forward* steam-pressure, P, against the *front* cylinder-head, so that $P - P' = R$, which after substitution gives us $P = \left(\frac{r}{a}\right)R$, the same as before.

If the wheels are not to slip, the horizontal action of the rail on the driver, viz., P'', Figs. 107 and 109, must not exceed the ultimate friction, F, or about one fifth the weight on the drivers. But $P'' = \frac{a}{r}P$, so that $P'' = R$; hence R should not exceed F.

89. Appold Automatic Friction-brake, or Dynamometer of Absorption.—Fig. 110 shows the *Appold* friction-brake, which is *automatic* by reason of its "compensating lever," BC. The brake is formed of a ring of wooden blocks, connected by an iron hoop, whose two ends are pivoted to the lever, as shown, at C. The pulley W revolves within. If the friction is insufficient to keep the weight G raised, it sinks and the end B of the lever presses against the fixed stop BK, thus tightening the hoop, increasing the friction, and raising G (and *vice versa* when the friction is too great).

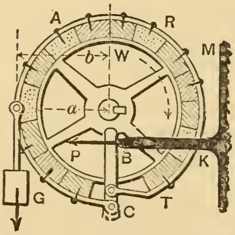

FIG. 110.

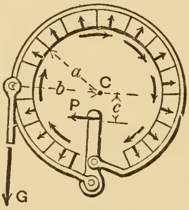

FIG. 111.

Fig. 111 shows the Appold brake as a free body. The normal pressures of the pulley against the wooden blocks have no moments about C, while the tangential actions (i.e., the frictions) have a common lever-arm, $= a$, about C. Hence, by equilibrium of moments, $Fa = Gb - Pc$, where F is the sum of the frictions. Evidently the pressure P at the end of the lever should be known, for accuracy.

(See correspondence in the *London Engineer*, from November, 1887, to March, 1888.)

Hence if v is the velocity (ft. per sec. for instance) of a point in the pulley-rim, the power absorbed is $L = Fv = \left(\frac{Gb - Pc}{a}\right)v$.

90. The "Carpentier" Dynamometer of Absorption. (Exhibited in Paris in 1880.)—This is automatic, like the Appold brake, but instead of automatically altering the tension in the strap to keep the weight "floating," it changes the arc of contact $A\,B$ until the friction is so altered ($=F'$) as to equilibrate the weight G' with help of the smaller weight G. See Fig. 112.

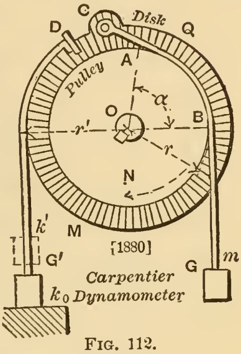

FIG. 112.

N is a pulley keyed upon the shaft of the motor to be tested, and therefore moving with it. A weight G is fastened to a strap BAC against which the pulley N rubs, but the upper end of this strap is attached to a block C, which is a rigid part of another pulley, or disk, M, *beyond* the pulley N. The disk M is *loose* on the shaft, the block C projecting over the face of the pulley N. The strap DG' is attached to the pulley M at D and carries a weight G' which we at first assume to be supported by a fixed platform k_0. When the shaft begins to turn (see arrow), and the tension in CA occasioned by the friction due to the arc AB and weight G is greater* than G', then G' will begin to rise, the disk M turning. But, as M turns, the uppermost point of contact of the strap AB on the pulley N moves to the right; i.e., the arc of contact AB becomes smaller, with a consequent reduction of the friction between N and the strap, so that after a little the tension in CA, pulling on C, is just sufficient ($=S'$) to keep the weight G' at rest at some point k'. When this state of equilibrium is reached, we have, by moments, for the equilibrium of the strap, $S'r = F'r + Gr$; and from that of the disk, $S'r = G'r'$; i.e., $F' = \dfrac{G'r' - Gr}{r}$, so that if $v =$ velocity of the pulley-rim, the power $= L, = Fv$,

$$= \left[\frac{G'r'}{r} - G\right]v, \text{ (ft.-lbs. per second, for instance.)}$$

91. Numerical Example in Boat-rowing.—As an illustration of the relations of the quantities concerned in a simple, but typical, case of propulsion on the water, let us suppose, in the problem of Fig. 166 of p. 161, M. of E., that the distances from the oar-handle, A, and oar-lock, C, to the middle of the blade are 9 ft. and 6 ft., respectively; and that a pull of 20 lbs. is exerted on A. The pressure at the oar-lock is then 30 lbs.; and that of the blade against the water, 10 lbs. Hence the boat is subjected to two forward oar-lock pressures of 30 lbs. each; to two backward pressures against the foot-rest, of 20 lbs. each; and to some backward resistance R of the water against the hull (R depending on the square of the velocity; $R =$ zero if there is no velocity).

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* Or, rather, its moment greater than that of G'.

The difference between the oar-lock and foot-rest pressures is a forward force of 20 lbs.; and if the velocity of the boat at the beginning of the stroke is such that R is $=$ 20 lbs., the effect is to barely maintain that particular speed while the 20 lbs. pressure is acting on each oar-handle, (and R remains constant also.) If R is smaller than 20 lbs. the velocity will be accelerated, and R will increase; if larger, the velocity diminishes (and R also), and of course will diminish at a much more rapid rate when the oar is lifted from the water.

For an ordinary small skiff (with pointed *stern* as well as bow) containing one person, the water-resistance R is (roughly) about one-half pound for each sq. foot of the wetted surface of the hull, *when the velocity is* 10 ft. *per sec.;* for other velocities, as the square of the velocity.

If in above case each oar-handle, while under the 20 lbs. pressure, passes through a distance $\overline{AE} = 3$ ft., *measured on the boat*, and the blade slips backward in the water an absolute distance s_1 of 6 inches (say), $= \frac{1}{2}$ ft., the absolute distance passed through by the boat will be (from the geometry of the figure) 5.5 ft. The work spent on slip is $2 \times 10 \times \frac{1}{2} = 10$ ft.-lbs.; so that, of the work, $2P \,.\, \overline{AE} = 2 \times 20 \times 3 = 120$ ft.-lbs., exerted by the oar-handle pressures, *relatively to the boat* (see fourth line of p. 161, M. of E.), 110 ft.-lbs. remain for the work of overcoming the resistance, R, and increasing the K. E. of the mass of boat. R is overcome through the distance $s_3 = \overline{CD}, = 5.5$ ft., so that if *all* of the 110 ft.-lbs. are spent on R (i.e., if the velocity is to be maintained constant), we must have $110 = R \times 5.5$, i.e., $R = 20$ lbs., as proved above. [In order that R may have this value with a small skiff the velocity must be (roughly) about 11 or 12 ft. per sec., at the beginning of stroke.]

Note that the absolute distance through which the two 20-lb. oar-handle pressures work is $5.5 + 3 = 8.5$ ft. But, of the absolute work done by them, viz., $2 \times 20 \times 8.5 = 340$ ft.-lbs., in the stroke, the amount $2 \times 20 \times 5.5$, $= 220$ ft.-lbs., is absorbed in overcoming the foot-rest pressures through 5.5 ft., the remainder being an amount $2 \times 20 \times 3$, $= 120$ ft.-lbs., $= 2P \,.\, AE$, to be spent on the work of slip, of liquid-resistance, and change of K. E. (if any).

92. Remarks on the Examples of § 155, M. of E.—In the first example the work to be computed is that done by the tension P, in the draw-bar, considered as a working force acting on the train behind the locomotive. If the velocity were uniform, a value $10 \times 200 = 2000$ lbs. would be sufficient for P; but as the velocity is to be increased, the "inertia" of the train is brought into play and the amount required is three times as great in this instance. In *Example* 2 P has the same significance.

As to *Example* 3, multiplying the 15,000 lbs. resistance by the speed reduced to ft. per sec. will give the power in ft.-lbs. per sec. Dividing this number by 550, we obtain 461 H. P. (see p. 136, M. of E.).

In *Example* 4 the resistance is greater than before, in the ratio of $(10)^2$ to $(15)^2$, i.e., it is $2\frac{1}{4}$ times 15,000 lbs. The distance through which it is overcome per second being $1\frac{1}{2}$ times its former value, we find the power spent on water-resistance to be (in ft.-lbs. per sec.) $2\frac{1}{4} \times 1\frac{1}{2} \times 15000 \times \dfrac{10 \times 6086}{60 \times 60}$; which divided by 550 gives 1556 H. P. (That is, the respective powers are as the *cubes* of the speeds.)

Example 5. Since 80 per cent of the power, L, of the working force (steam-pressure) is to be 461 H. P., we write $0.80 \times L = 461$, and obtain $L = 576$ H. P.

Example 6. If the thrust or pull of the connecting-rod of the engine on the crank-pin be resolved at every point into a tangential and a normal component, T and N (Fig. 113), we note that T is a working force and N neutral. Hence at this point in the line of transmission of power we can ascribe all the power to the force T. T is variable, and by its average value, T_m, we mean a value whose product by the length of the circumference described by the crank-pin shall be the same amount of work as that actually done by the variable

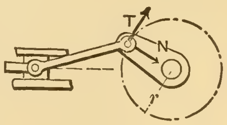

FIG. 113.

T per revolution. Now 461 H. P. means 253,000 ft.-lbs. of work per second, which divided by 1.0, the number of turns per sec., gives the work done by T in each turn. Hence $T_m 2\pi \times 1.5$ should $= 253,000$; or $T_m = 26,890$ lbs.

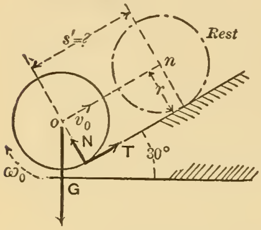

Fig. 114.

Example 7. Fig. 114. At O we have the sphere in its initial position, the forces acting on it being its own weight G (a resistance), and the two components, N and T, of the pressure of the inclined plane against it. Since there is no slipping (i.e., perfect rolling), both N and T are *neutral* (see § 84).

Let $v_0 =$ initial velocity of the centre of gravity, and ω_0 the initial angular velocity. $s' \sin 30°$ is the unknown height through which the centre will rise before the velocity becomes zero.

The initial K. E. of translation is $\frac{G}{g} \cdot \frac{v_0^2}{2}$; and of rotation, $\frac{1}{2}\omega_0^2 M k_0^2$. Now $v_0 = \omega_0 r$ and $k_0^2 = \frac{2}{5}r^2$ (p. 102). Hence by eq. (XV), p. 147,

$$-Gs' \cdot \tfrac{1}{2} = 0 - G\left[\frac{3 \times 3}{2 \times 32.2}\right](1 + \tfrac{2}{5}); \quad \text{or} \quad s' = 0.39 \text{ ft.}$$

93. Efficiency of a Wedge. Block on Inclined Plane.—In the numerical example 3 of p. 172, M. of E., it is to be noted that the "efficiency" of the mechanism, or ratio of the work usefully employed in overcoming Q, to that exerted by the one working force P, is $57.6 \div 153.6 = 37.5$ per cent.

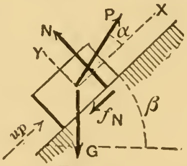

Fig. 115.

Fig. 115 shows as a free body the block mentioned in problem 6, p. 172, M. of E. We are to find the force P in the given action-line, such that a *uniform* motion (equilibrium) can be maintained *up* the plane. The action of the plane on the block is represented by the normal pressure N and the tangential action, or friction, fN. [The student should not assume thoughtlessly that N must $= G \cos \beta$ (the normal component of G), for the unknown P, not being parallel to the plane, has an influence in determining N; here it tends to relieve the pressure on the plane.] The forces being balanced, ΣX, and ΣY, $= 0$;

whence $$P \cos \alpha - fN - G \sin \beta = 0;$$

and $$P \sin \alpha + N - G \cos \beta = 0;$$

and finally $$P = \frac{G[\sin \beta + f \cos \beta]}{\cos \alpha + f \sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Problem 7 calls for the value of P if the motion is to be *down* the plane; other things as before. Fig. 116 shows the change. The friction acts in a contrary direction; P will be smaller; N, larger. Solving as before, we finally have

$$P = \frac{G[\sin \beta - f \cos \beta]}{\cos \alpha - f \sin \alpha} \quad . \quad . \quad . \quad (2)$$

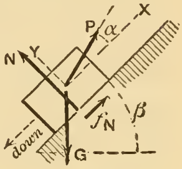

FIG. 116.

We note here [eq. (2)] that for $\beta = \phi =$ the angle of friction, $P = 0$; and that if f is large and β small ($< \phi$), P may be negative, i.e., its direction may need to be reversed, to *aid* the body down the plane.

94. Work Absorbed in Rolling Resistance.—With perfect rolling of a smooth rigid wheel upon a straight, fixed, smooth, and *incompressible* rail, the pressure R of rail on wheel is a neutral force as regards work (call this "*perfect rolling*"); for its point of application, at the foot of the perpendicular let fall from the centre of wheel on the rail, is *motionless* so long as it *is* the point of application. But with an inelastic, compressible rail, the point of application is a little ahead (distance $= b$) of the foot of that perpendicular, and it results, therefore, that for every ds moved through by the centre of the wheel the rail-pressure is overcome through the small distance $\left(\frac{b}{r}\right)ds$; so that when the wheel-centre (or any point of the car-frame) is passing through the distance s (parallel to rail), the work done on the rail-pressure, R, is not zero, but $= R \cdot \frac{b}{r}s$. It is convenient, therefore, in applying the principle of work and energy to the case of a car, to consider that the rolling is *perfect* and that the work actually due to rolling resistance is occasioned by the overcoming of an imaginary backward force, $\frac{b}{r}R$, applied directly to the car-frame, or centre of axle of wheel; since the product $\left[\frac{b}{r}R\right]s$ is equal to that of R by $\left(\frac{b}{r}s\right)$. This imaginary force is what is referred to in the foot-note of p. 189, M. of E.

95. Examples of § 173, M. of E.—In Example 2, since slipping is impending, the backward tangential action, or force, of the rail on each wheel must be T, $= (0.20 \times 5000)$ lbs.; and since its moment, Tr, about the wheel-axis must balance that, Fr, of the friction F between brake and wheel-rim (neglecting the moment of the small axle-friction), this friction must be $F = 1000$ lbs.

When the car-frame moves through the unknown distance s before coming to rest, the friction F (an internal friction) is overcome through a relative distance measured on the wheel-tread of length $= s$ (because the wheels do not slip and the brake-shoe presses on the *same* rim as that which rolls on the rail; whereas, if the shoe rubbed on the rim of a drum on the same axle, the circumference of the drum being (for example) one half that of the wheel-rim, the relative distance rubbed through would only be one half of s.

Gravity being neutral and both axle and rolling resistances being neglected, we have, from eq. (XVI), p. 149, M. of E., (with foot, pound, and second,)

$$0 - 8 \times 1000 \times s = 0 - \frac{1}{2}\frac{40000}{32.2}(80)^2; \quad \therefore\ s = 496 \text{ ft.}$$

In *Example* 3, although the track is on an up-grade, or inclination α, with the horizontal, the angle is so small that $60 \div 5280$ may be taken as either its nat. tang. or its sine, at will. The weight 40,000 lbs. is here a resistance, being raised through a height of 1000 ft. × sin α, = 11.35 ft., and both rolling and axle resistances are neglected. Hence

$$0 - 8F \times 1000 \text{ ft.} - 40000 \text{ lbs.} \times 11.35 \text{ ft.} = 0 - \frac{1}{2} \cdot \frac{40000}{32.2}(80)^2;$$

or $F = 439$ lbs.

Example 4 differs from the preceding only in the introduction of two more items of negative work. The work of rolling resistance is ascribed to the overcoming of an imaginary force $= \left(\frac{.02}{15} \times 5000\right)$ lbs., = 6.66 lbs. for each wheel (applied to car-frame), through a distance of 1000 feet.

The axle-friction at the journal of each wheel is (0.036×5000) lbs. = 180 lbs. (taking the coefficient near the top of p. 190, M. of E.). A point in the circumference of the journal rubs through a distance of $\left(\frac{1.5}{15}\right) \times 1000$ ft. = 100 ft. Hence, finally,

$$-8F \times 1000 - 40000 \times 11.35 - 8 \times 6.66 \times 1000 - 8 \times 180 \times 100$$

$$= 0 - \frac{1}{2} \cdot \frac{40000}{32.2}(80)^2; \quad \text{i.e.,} \quad F = 414 \text{ lbs.}$$

96. Examples in Dynamics. (Statements on p. 194, M. of E.) —*Example* 1. Fig. 117 shows the end of the shaft in question. The friction on the side, or "axle friction," is due to the pressure of 40 tons. $= N'$; i.e., $F' = fN' = .05 \times 80{,}000 = 4000$ lbs., and is overcome through a circumference of $\pi \times 1$ ft., = 3.14 ft., each revolution. The work done per revolution on the friction at the base (due to the 10 tons) is the same as if all concentrated at a distance $= \frac{2}{3}$ of the radius from the centre, and is therefore $(.05 \times 20000) \times (\frac{2}{3}\pi \times 1) = (1000 \text{ lbs.}) \times (2.094 \text{ ft.})$. Hence the lost work per second, i.e., the *power*, absorbed by friction, *in ft.-lbs. per sec.*, is

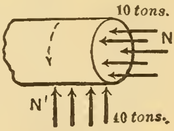

FIG. 117.

$$\frac{50}{60}[4000 \times 3.14 + 1000 \times 2.094] = 12211 = 22.2 \text{ H. P.}$$

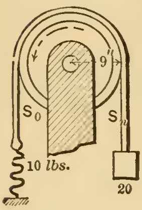

FIG. 118.

Example 2. Fig. 118. (This form of friction-brake is sometimes used in testing motors.) Since slipping *actually occurs*, we use eq. (3) of p. 183, M. of E., $S_n = S_0 e^{f\alpha}$; i.e., $f\alpha = \log_\epsilon \left[\frac{S_n}{S_0}\right]$. Here we put $S_n = 20$ lbs., $S_0 = 10$ lbs., with $\alpha = \pi$, and obtain

$$f = [2.302 \times \log_{10}(2.0)] \div \pi, = 0.22.$$

As to the power, L, expended on the friction, we note that each element of the friction is overcome through a distance v every second, where v is the linear velocity of the pulley-rim; i.e., $L = Fv$; denoting the sum of the elementary frictions by F. These elementary frictions are not parallel, but we note that each (being tangential) has the same lever-arm, = 9 inches, $= r$, about the centre of the circular curve, so that when the curved part of the strap is considered free, and moments are taken about that centre, we obtain $Fr = S_n r - S_0 r$ (the normal pressures on the cord have zero moments); hence $F = S_n - S_0$ and the power in question $= L = [S_n - S_0]v$. Therefore

$$L = [(20 - 10) \text{ lbs.}] \times [(2\pi \times \tfrac{9}{12} \times \tfrac{100}{60}) \text{ ft. per sec.}] = 78.5$$

ft.-lbs. per sec.; = 0.142 H. P.

Should the pulley be made to turn the other way with the same speed, the weight of 20 lbs. (now $= S_0$) remaining the same, more power will be required. Assuming the coefficient of friction between strap and pulley to remain unchanged, the spring-balance will now read 40 lbs. tension ($= S_n$), since $e^{f\alpha} = 2$, so that $S_n - S_0 = 20$ lbs. instead of 10 as before. v being the same (7.85 ft. per sec.), the power will be double that previously found; viz., L now = 0.282 H. P.

Example 3. Fig. 119. It is assumed that the pressure on the journals, or axle, is due solely to the weight, G, of the stone; i.e., that the caps of the bearings are not clamped in contact with the journals; otherwise the friction would be $> f'G$, for axle-friction.

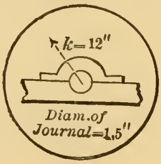

FIG. 119.

The stone being considered free, we note that its weight and the *normal* component of the reaction of the bearing are neutral forces, the friction, or tangential component, $f'G$, is a resistance; and that there are *no* working forces.

The initial K. E.* of the stone is $\frac{1}{2}\omega_0^2\frac{G}{g}k^2$ (ω_0 being the initial angular velocity); its final K. E., zero. Hence the initial K. E. is all absorbed in the work of friction. Since the number of turns in coming to rest is 160, the total distance through which the friction at the circumference of the journal is overcome is $160 \times \pi[1.5 \div 12] = 62.85$ ft. With the *foot* and *second*, $\omega_0 = 2\pi \times \frac{120}{60} = 4\pi$ radians. and $g = 32.2$. Therefore. from the *Principle of Work and Kinetic Energy* [eq. (XV), p. 147, M. of E.],

$$\tfrac{1}{2}(4\pi)^2\frac{G}{32.2}\left(\frac{12}{12}\right)^2 = f'G \times 62.85; \quad \text{and} \therefore f' = 0.039.$$

Example 4. To move A horizontally with an acceleration $= 15$ (foot and sec.) would require a horizontal force $=$ *mass* $\times$ *acc.* $= \frac{2}{32.2} \times 15 = .93$ lbs. But the horiz. compon. (friction) of the action of B on A is only $= fN = 0.3 \times 2 = 0.6$ lbs., which is $< .93$. Hence A will not keep abreast of B, but will gradually fall behind B.

97. Brake-strap, Lever, and Descending Weight. Numerical Example (Fig. 120).—The weight Q of 600 lbs. is to be let down without acceleration, the rope unwinding from a drum of 1 ft. radius. On the shaft of the drum is keyed a pulley of 2 ft. radius, the friction on whose rim, due to its rubbing under the encircling stationary strap, can be varied in amount by a force P exerted on the lever AB. It is required to compute a proper value for the pressure P in the present instance to *prevent acceleration* of the 600-lb. weight, the coefficient of friction of the strap on the pulley being assumed to be $f = 0.30$.

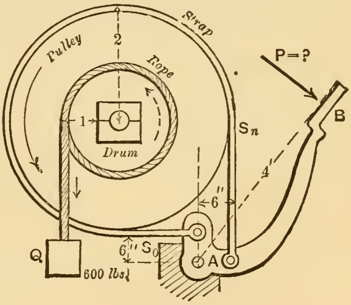

FIG. 120.

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* Kinetic Energy.

The strap covers three quarters of the pulley-rim (i.e., $\alpha = \frac{3}{2}\pi$). See figure for other dimensions.

If Q sinks without acceleration, the tension in the vertical part of the rope must be 600 lbs., and the rotation of pulley be uniform; hence moments must balance for the pulley, drum, and shaft; so that (with the foot, pound, and second) $600 \times 1' = F \times 2'$, where F is the sum of the requisite frictions (tangential forces) of strap on pulley; i.e., $F = 300$ lbs.

But from the equilibrium of the curved portion of strap, by moments about centre of the curve, $S_0 \times 2' - S_n \times 2' + F \times 2' = 0$. S_n is the tension in the vertical straight part of the strap; S_0, that in the horizontal. S_n is greater than S_0 and bears to it the relation $S_n = S_0 e^{f\alpha}$; or, $\frac{3}{2} f\pi = \log_\epsilon \left(\frac{S_n}{S_0}\right)$. That is (see p. 184, M. of E.), $(S_n : S_0) =$ the number whose common logarithm is $(.45)\pi \times 0.43429 = 4.12$. Combining this with $F = S_n - S_0$ (see above), we have $(4.12 - 1)S_0 = F = 300$ lbs.; whence $S_0 = 300 \div 3.12 = 96.15$ lbs.; and $S_n, = 4.12S_0, = 396.13$ lbs.

The requisite force P is then found by noting that for the equilibrium of the lever, the moments of the three forces S_n, S_0, and P, must balance about the fulcrum A; i.e.,

$$P \times 4' = S_n \times \tfrac{1}{2} + S_0 \times \tfrac{1}{2}; \quad \text{or} \quad P = \tfrac{1}{8}(S_n + S_0) = 61.5 \text{ lbs.};$$

a pressure easily applied by one man.

CHAPTER VII.

MECHANICS OF MATERIALS AND GRAPHICAL STATICS.

98. Intensity of the Shearing, and of the Normal, Stress on an Oblique Section of a Prism under Tension. (This treatment may be more readily understood than that now given in § 182 of M. of E.)—From the prism under tension in Fig. 193, M. of E., consider by itself a portion shown in Fig. 121, between the right section *KN* and any oblique section, *ML*. The area of the right section being *F* and the intensity stress per unit area of that section being *p*, *Fp* expresses the total stress on the right section. The total stress on *ML* is also of course $= Fp$. Its component P'' normal to the plane *ML* is evidently $P'' = Fp \sin \alpha$. This is the total normal stress on this oblique plane *ML*. But the area *ML* over which this normal stress is distributed is not $= F$, that of the right section of the prism, but $= (F \div \sin \alpha)$. Hence to obtain the normal stress on *ML per unit of area*, i.e., the *intensity* of normal stress on *ML*, we must divide P'' by $(F \div \sin \alpha)$ and thus obtain:

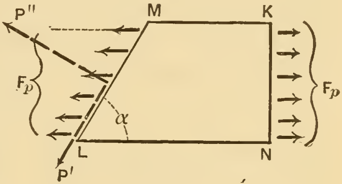


FIG. 121.

$$\left.\begin{array}{r}\textit{Normal stress, per unit of}\\ \textit{area, on oblique section}\end{array}\right\} = p \,.\, \sin^2 \alpha. \quad . \quad . \quad (1)'$$

Similarly, the other component, P' (of the total stress on *ML*), which is tangential to *ML*, is in amount $= Fp \cos \alpha$, which is the total shearing stress on *ML*. To obtain the *intensity* of this shearing stress, we divide P' by the area ($F \div \sin\alpha$) of ML, over which P' is distributed, and obtain:

$$\left.\begin{matrix}\textit{Shearing stress, per unit}\\ \textit{of area, on oblique section}\end{matrix}\right\} = p \,.\, \sin\alpha\cos\alpha. \qquad . \quad (2)'$$

In these two equations p is an abbreviation for $P \div F$, and α is the angle that the oblique plane ML makes with the axis of the prism.

The reason for ascertaining the stress *per unit area* in any case is, of course, that the safety of the material depends upon it and not simply on the total stress.

99. Spacing of Rivets in a Built Beam.—The statement in the middle of p. 293 of M. of E. that "The riveting connecting the angles with the flanges (or the web with the angles), in any locality of a built beam, must safely sustain *a shear equal to J* (the total shear of the section) *on a horizontal length equal to the height of the web*," may be most directly utilized as follows: Imagine the horizontal continuity of the web to be broken and consequently a vertical seam rendered necessary, as shown in Fig. 122.

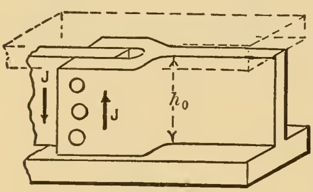


FIG. 122.

Then, whatever spacing of rivets would be necessary in this *ideal* seam can be adopted in the real horizontal seam made by riveting together the web and angles, the rivets being considered to be in double shear (or single) in the ideal case, if so in the actual. *For example*, taking the data of the example on p. 294, M. of E., there must be enough rivets in the vertical seam of length $= h_0 =$ the height of web, to safely stand the total shear of $J = 40{,}000$ lbs. Since each rivet can safely endure a shear of 9000 lbs. in double shear (see p. 294), the number of rivets required in the height of web would be $40000 \div 9000 = 4.44$; i.e., they should be spaced 4.5 in. apart since the height of web is 20 in., and $20'' \div 4.44 = 4.5''$.

But since the pressure on the side of each hole is limited to [illegible]470 lbs. (see p. 294), on this basis the number of rivets in height of web should be $40{,}000 \div 5470 = 7.2$, which implies putting them at a distance apart, centre to centre, of $20 \div 7.2 = 2.7$ in. This spacing, therefore, should be adopted in the horizontal seam between web and angles at this part of the beam (near abutment).

100. I-beams treated without the Use of the Moment of Inertia.—The assumption is so frequently made (for simplicity in treating the web) that the web carries all the shear, that the corresponding assumption that the two flanges carry all the tension and compression is also sometimes used to simplify the analysis. With thin webs the results obtained are accurate enough for practical purposes. For example, suppose a horizontal I-beam to rest upon supports at its extremities and to bear several detached loads. To find the tensile or compressive stress in the flanges at any right section as m, consider free the part of beam on the left of that section (see Fig. 123). R is the abutment reaction; P_1 and P_2 the loads between m and A. Considering the total compression P' to be uniformly distributed over the area of the upper flange, and hence (for geometrical purposes) to be applied in the centre of gravity of that flange, and similarly the total tension P'' in the centre of gravity of the lower flange (h' = vertical distance between), while the web carries the shear alone; we obtain (by taking moments about the intersection of J and P''),

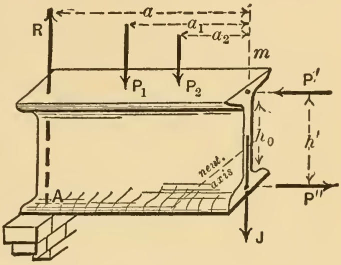


FIG. 123.

$$P_1a_1 + P_2a_2 + P'h' = Ra;$$

from which P' is found. Evidently $P'' = P'$, since they are the only forces having horizontal components. Let F' = the area of either flange, then the stress per unit area in either flange is $p' = \dfrac{P'}{F'}$. If this result is too large for safety the flange-area must be increased.

By this method, the proper amount of sectional area can be computed for the flanges at each of several sections of a beam to carry a specified loading, and a beam of "uniform strength" be designed. In such a case, however, the weight of the beam itself cannot be estimated in advance, but after a gradual tentative adjustment can be taken into account in a satisfactory manner.

(The upper flange being in compression may need to be braced or latticed with those of the accompanying parallel beams to prevent horizontal buckling.) Plate girders of variable section are designed on this general principle, the flange-area being increased toward the middle of the span by riveting on additional plates.

101. Load of "Incipient Flexure" of Long Slender Columns.—The result brought out in eq. (8) of p. 366, M. of E., that in the case of a long, *slender*, round-ended, prismatic column, no flexure at all takes place until a definite value for the load is reached, and that, with that value, any small deflection whatever can be maintained, may be arrived at quite rationally as follows, without intricate analysis:

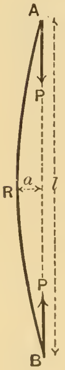


FIG. 124.

Let the horizontal bed-plates of a testing machine, Fig. 124, be advanced toward each other until the slender round-ended rod AB between them is deflected a small amount $= a$ at the middle. If ρ is the radius of curvature of the elastic curve at R, we have (from free body $R\,.\,.\,A$)

$$\frac{EI}{\rho} = Pa; \quad \text{or} \quad P = \frac{EI}{a\rho}.$$

(This radius of curvature, ρ, is nearly = to that of the circle determined by the three points B, R, and A ($\frac{8}{10}$ of it).

Now let the plates be separated until the deflection is only $= a'$; call the new radius of curvature, ρ'. Then the new value of the force or pressure at each end is $P' = \frac{EI}{a'\rho'}$. But, for this very flat curve, as the deflection is diminished, the radius of curvature changes in an inverse ratio; i.e., $\rho' : \rho :: a : a'$; or, $a\rho = a'\rho'$. Therefore $P = P'$; i.e., *the pressure induced by the elasticity of the rod against the plates does not diminish with a diminishing deflection, but remains constant.* Or, conversely, if a less force than this is applied to the column, no deflection takes place; while if an actual load, whose weight is greater than the above force, be placed on the column, the upward pressure against it due to the elasticity of the column (as the latter bends) being always less than the weight (unless perhaps when the deflection becomes extreme), the load sinks with an accelerated motion and the column finally breaks (since with increased deflection the stress in the outer fibre is augmented).

In most cases in practice, columns are not sufficiently slender to bear out all the above-described phenomena, but enough has been said to show that while with a horizontal beam the deflection is nearly proportional to the load applied (within elastic limit), such is far from being the case with a column.

If a piece of card-board cut from a visiting card, and about half an inch wide by three or four inches long, be pressed endwise with the finger on the scale-pan of a letter-scale, the value of the pressure corresponding to different deflections can be easily noted and the above claims roughly verified.

102. Recent Tests of Large Wooden Posts.—Prof. Lanza, of the Massachusetts Institute of Technology, Boston, has made tests on yellow-pine flat-ended posts, giving results as follows: Highest breaking stress 5400 lbs. per sq. in.; lowest, 3600; average, 4544.

These yellow-pine posts were nearly cylindrical in form and almost all of them 12 ft. in length (a few 2 ft. long); diameters, from 6 to 10 in.

With *white oak* posts, flat-ended, and of about the same sizes as the former, the highest breaking stress was 4600, the lowest, 3000 lbs. per sq. in.; with exception of two which reached 6000. In all these cases of pine and oak posts failure occurred by direct crushing, lateral deflections being inconsiderable, showing that all were practically "*short blocks.*"

Eight separate tests were made with the load applied eccentrically, about two inches off the centre, the result being to diminish the strength by about one third. All these posts had been in use for years and were well seasoned. Each had a hole about 2 in. in diam. along the axis, from end to end.

Other tests have been made at Watertown, Mass., with the

Government testing machine on timber columns, of rectangular sections, mostly about 5 by 5 in., and 7.5 by 9.7 in.; with a number 5 by 15 in., and 8 by 16 in. Their lengths ranged progressively from 15 in. to 27 ft. Flat-ended supports. From these tests Mr. Ely concludes that if the breaking load in *pounds* be expressed as $P = FC$, where F is the sectional area in square inches, one of the following values for C should be taken according to the kind of timber and the ratio of the length l to the least side, b, of the rectangular section; thus:

For *white pine:*

For $l \div b = 0$ to 10; 10 to 35; 35 to 45; 45 to 60,
Make $C =$ 2500; 2000; 1500; 1000.

For *yellow pine:*

For $l \div b = 0$ to 15; 15 to 30; 30 to 40; 40 to 45; 45 to 50; 50 to 60,
Make $C =$ 4000; 3500; 3000; 2500; 2000; 1500.

103. The Pencoyd Tests of Full-size Rolled-iron I-beams, Channels, Angles, Tees, etc., used as Columns.—These were made in 1883 by Mr. Christie, at the Pencoyd Iron Works of Philadelphia, Pa., and were very careful and extensive. The following table is based on them. By "fixed ends" it is here meant that the ends are so securely attached to the contiguous supports that the fastenings would not be ruptured if the column were subjected to a breaking load; by "flat ends," that the ends are squared off and bear on a flat, firm surface. "Hinged ends" refers to the ends being fitted with pins, or ball-and-socket joints, of proper size, with centres practically in the *axis of the column* (this axis being the line containing the *centres of gravity of all sections* of the column, which is prismatic); while "round ends" implies that the ends have only points of contact such as balls or pins bearing on a flat plate, the point of contact being in the axis of the column. The first column of the table contains the ratio of the length, l, of the column to k, the least radius of gyration of the section (except that for hinged ends the pin must be at right angles to the least radius of gyration). Factors of safety are recommended as given, being different for fixed and flat ends

Ratio $\frac{l}{k}$	Fixed Ends.	Flat Ends.	Factor of Safety.	Hinged Ends.	Round Ends.	Factor of Safety.
20	46,000	46,000	3.2	46,000	44,000	3.3
40	40.000	40,000	3.4	40,000	36,500	3.6
60	36.000	36,000	3.6	36,000	30,500	3.9
80	32.000	32,000	3.8	31,500	25,000	4.2
100	30,000	29,800	4.0	28,000	20,500	4.5
120	28,000	26,300	4.2	24,300	16,500	4.8
140	25,500	23,500	4.4	21,000	12.800	5.1
160	23,000	20,000	4.6	16,500	9,500	5.4
180	20,000	16,800	4.8	12,800	7,500	5.7
200	17,500	14,500	5.0	10 800	6,000	6.0
220	15,000	12,700	5.2	8,800	5,000	6.3
240	13,000	11.200	5.4	7,500	4.300	6.6
260	11,000	9,800	5.6	6,500	3,800	6.9
280	10,000	8.500	5.8	5,700	3,200	7.2
300	9,000	7,200	6.0	5,000	2,800	7.5
320	8,000	6,000	6.2	4,500	2,500	7.8
340	7.000	5,100	6 4	4,000	2,100	8.1
360	6,500	4,300	6.6	3,500	1,900	8.4
380	5.800	3,500	6.8	3,000	1,700	8.7
400	5,200	3.000	7.0	2,500	1,500	9.0
420	4,800	2,500	.....	2,300	1,300	.....
440	4,300	2,200	.....	2,100	......	.....
460	3,800	2,000	.....	1,900	......	.....
480	......	1,900	.....	1,800	......	.....

from those proposed for hinged and round ends. In the other columns the numbers given are the respective breaking stresses in lbs. per *sq. in. of sectional area*, which number must be multiplied by that area in sq. in., for the actual breaking load; dividing which by the proper factor of safety we obtain the safe load in pounds.

For example, required the breaking load of a 9-in. light iron I-beam of the N. J. Steel and Iron Co., 14 ft. long and used as a column with flat ends.

From p. 40 of the hand-book we find: least $I = 4.92$ and the area of section $= 7$ sq. in. Hence the least k^2 is $I \div F$, $= 0.703$, and k itself $= 0.838$ in.; so that $l \div k = 168$ in. $\div$ 0.838, $= 200$. The breaking load, then, is $FC = 7.00 \times 14,500$ $= 101,500$ lbs.; and the safe load would be $101,500 \div 5 = 20,300$ lbs. (If Rankine's formula were applied to this same case a fair agreement with this result would be found.)

104. Cast-iron Channel as a Column.—In the case of the channel-shaped section in Fig. 125, composed of three rectangles, all of the *same width*, $= t$, it is desirable, for economy of material if the prismatic body is to be used as a column, that the moments of inertia about the two gravity axes X and Y should be equal (see § 311, M. of E.). The problem, therefore, presents itself in this form: Given the width b of the base DC of the section, and the thickness of metal t, what value should be given to the length c, of the projecting sides AD and BC, of the channel to secure this result?

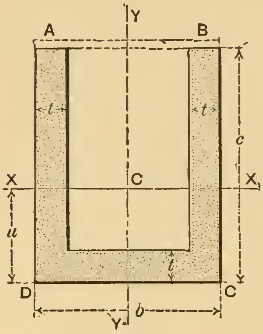


FIG. 125.

The algebraic statement of the conditions involved leads to an equation of high degree for the unknown quantity c. But, *by numerical trial*, a few reliable values have been found, given below, by simple interpolation between which all ordinary cases in practice may be satisfied.

When $t = 0$ (infinitely thin), make $c = 1.37b$;
" $t = 0.083b = \frac{1}{12}b$, " $c = 1.30b$;
" $t = 0.166b = \frac{1}{6}b$, " $c = 1.25b$;
" $t = 0.500b = \frac{1}{2}b$ $\left\{ \begin{array}{c} \text{solid} \\ \text{rectangle} \end{array} \right\}$ " $c = 1.00b$.

In the construction of such a column the edges A and B of the projecting sides should be tied or braced together at intervals; or occasional transverse webs may be introduced.

It is remarkable in this problem that the distance u of the centre of gravity C from the base DC is *almost exactly equal to one half of b in every instance*, and may be so assumed in locating the axis of the column that the load may be applied in that axis.

105. Vertical Reactions of Horizontal-faced Piers bearing a Beam and Loads. Graphical Method.—The construction on p. 404 of M. of E. is the one usually given and is the most convenient; still, the proof is a little puzzling to the student because the amounts and direction of the two auxiliary, mutually annulling, forces P and P' are not known at the outset.

Hence we here present a construction in which those two forces are completely assumed and known at the outset. Assume 1 and 6 as the auxiliary forces (equal, opposite, and in the same line), there being three loads in this instance, 2, 3, and 4; Fig. 126. Number the forces of the system as in figure and draw a portion of the force-polygon with the known forces 1, 2, 3, and 4, beginning at O, and the first three rays, I, II, and III (dotted). The first segment of the corresponding equilibrium polygon should begin at a, the intersection of the action-lines of forces 1 and 2. and finally the third segment cuts the action-line of 5 in the point m. Now the fourth segment is the last in this case [of three loads, 2, 3, 4], and must pass through the intersection of the last two forces, 6 and 7, i.e., through A. Hence draw mA and a parallel to it through the pole O, this latter line being the fourth ray desired, whose intersection with the "load-line" at n' cuts off the proper length of the right-hand reaction 5. The forces 6 and 7 are now easily added to the force-diagram in an obvious manner and the latter is complete; the magnitude of the other reaction, 7, being thus determined. Of course, the value of force 7 is also given by the $n'd$, and the force-polygon could also be closed by running from n' to d and then from d to O, instead of in the manner shown.

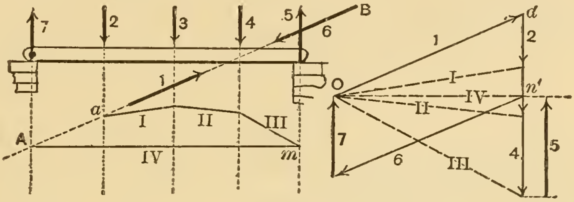


FIG. 126.

Note the order of numbering in the above. The two assumed forces are made the *first* and the *last but one*, respectively; the unknown reactions are made the *last but two*, and the *last*, respectively; while the given loads are assigned to the intervening numbers, in any order.

106. Construction for Use with the Treatment in § 390, M. of E. (Graphical Statics).—For the particular case involved in the treatment of the straight horizontal girder, built in, of § 390, M. of E., the following is given, to replace the more general construction of § 377, M. of E.

At (I.) in Fig. 127, we have a curve, or broken line (equilibrium polygon, for instance), $FKWMG$, connecting points F and G in two vertical lines. Across this curve we wish to draw a right line $v \,.\,.\, m$, in such a way that the area KWM above $v \,.\,.\, m$ shall be equal to the sum of the areas vFK and mGM below $v \,.\,.\, m$, and *also* that the centre of gravity of the upper area shall be in the same vertical line as that of the two lower, or *negative* areas, taken together.

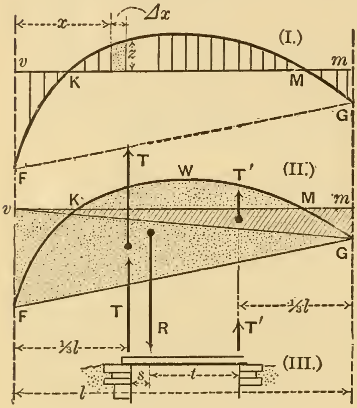


Fig. 127.

Only one position of $v \,.\,.\, m$ will do this, the algebraic expression for which is that $\Sigma(z') = 0$, and that $\Sigma(xz') = 0$; (the areas in question being divided into vertical strips of equal horizontal widths $= \Delta x$, the distance of any strip from the vertical line vF being called x.)

Annex the figure $FGMK$ (having joined $F \,.\,.\, G$) to both the positive and negative figures above mentioned and the condition now becomes [see (II.) in figure] that the area of the figure $FKWMG \,.\,.\, F$ (the right line FG being its lower boundary) must equal that of the two triangles vFG and mGv, and that the centre of gravity of the former must lie in the *same vertical* as that of the two triangles combined. In other words, if the area of the curvilinear figure, $FKWMG$-F, with FG as base, be considered as a weight R acting through its centre of gravity, then the areas of the two triangles must represent the two *upward reactions* T and T' of two piers [see (III.) in figure] supporting a horizontal beam on which R rests. These imaginary piers are evidently at distances of one third the span from the verticals through *F* and *G*. Adopt, therefore, the following construction:

By dividing into vertical strips find the area of the curvilinear figure with base *FG*. Draw the pier verticals at the one-third points. Find the vertical containing the centre of gravity of the curvilinear figure by p. 415. Compute or construct the values of T and T' on the conception of the known area R being a weight supported on the beam with T and T' as reactions. (T and T' are most easily obtained, perhaps, by scaling the distances s and t (see figure) and writing $T(s+t) = Rt$, and $T' = R - T$.)

If the area of the triangle *FGv* must be $= T$, one of the values just found, knowing that this area $= \frac{1}{2}$ (*altitude* l) $\times$ base $\overline{vF}$, the proper length of *vF* is easily computed; and similarly, using T' and the triangle *mvG*, we calculate $\overline{mG}$. Joining v and m, the required right line *vm* is determined.

107. Three-point Construction. Equilibrium Polygon for Non-vertical Forces. Preliminary Step.—In the construction and proof of § 378*a*, M. of E., it is supposed at the outset that the action-line of the resultant (R_1) of all the forces acting between points *A* and *p* has been found; similarly, that of the resultant (R_2) of the forces acting between *p* and *B*; and that of the resultant (R) of the two partial resultants.

It is here intended to give the detail of finding these three lines and to make clearer the scope and intent of the problem. Fig. 128. Let *A*, *p*, and *B* be the three points through which the equilibrium polygon is to pass, and 1, 2, etc. (to 6 inclusive) (on left of figure), the given forces in magnitude and position; 1, 2, and 3 acting between *A* and *p*; and 4, 5, and 6, between *p* and *B*. Lay off the "load-line" *STU*, on some convenient scale of force, and select any pole as O'''. Draw the "rays," $O'''S$, etc.; and also lines parallel to them, in proper order, beginning at a convenient point A'' (not *A*, necessarily), so as to form an equilibrium polygon, $A''B''$, as shown. The first segment is parallel to $O'''S$; the last, to $O'''U$; and the segment connecting forces 3 and 4 (in this particular instance), parallel to $O'''T$. The intersection of the first and last segments gives M'', a point in the action-line of R, the resultant of all the given forces, 1 . . 6; and similarly, the other intersections N'' and O'' are points in the action-lines of R_1 and R_2, respectively. Right lines through these three points, parallel respectively to SU, ST, and TU, should meet in a common point C', and are the respective action-lines desired.

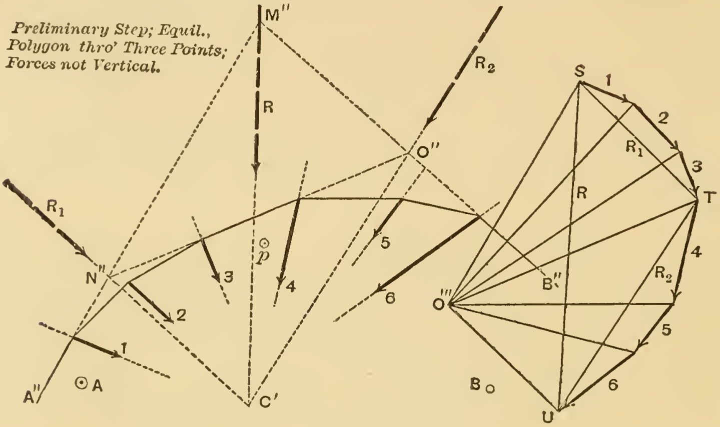

Preliminary Step; Equil., Polygon thro' Three Points; Forces not Vertical.
M''
R
R₂
S
1
2
R₁
3
T
O''
R
R₁
5
4
p
3
4
N''
2
R₂
B''
6
O'''
1
5
A''
A
B
6
C'
U

FIG. 128.

The further steps are those given on pp. 460 and 461, M. of E. Our present equilibrium polygon, $A''B''$, is of no further use; but the load-line STU will still serve, after the lines $M'A$ and $M'B$ have been located according to § 378a. We can then draw through S and U parallels to $M'A$ and $M'B$, respectively, and by the intersection of these parallels with each other determine the pole corresponding to the final equilibrium polygon which is to pass through A, p, and B.

It is to be noted that the problem of the "Shear-legs" of § 59 of these Notes, and also Problem 2 (of the two links), p. 35, M. of E., are cases of a three-hinged arch-rib, and can be treated graphically in the same manner; and thus the "special" equilibrium polygon and its corresponding pole and rays (i.e., force-diagram) determined.

The ray parallel to the segment passing through the intermediate joint (p) gives the amount and direction of the pressure on the hinge of that joint; and corresponding statements may be made for the two extreme joints.

CHAPTER VIII.

MISCELLANEOUS NOTES.

108. Co-ordinates of Centre of Gravity—Fuller Explanation.— Assume the various small particles of a rigid body to be numbered 1, 2, 3, etc., and call their respective volumes dV_1, dV_2, dV_3, etc. (cubic feet), and their x-co-ordinates x_1, x_2, x_3, etc. (feet). If the body is heterogeneous, the particles may be of different densities; for example, particle 1 may be of such a density that a cubic foot of material of that density would weigh 100 lbs., while a cubic foot of the material of which particle 2 is composed would weigh 110 lbs.; and so on. Or, in symbols, the "heaviness," or rate of weight, of particle 1 is $\gamma_1 = 100$ lbs. per cubic foot, while that of particle 2 is $\gamma_2 = 110$ lbs. per cubic foot. A similar notation would apply to all the other particles.

The respective weights, then, of the particles (or force of the earth's attraction on them) are $\gamma_1 dV_1$, $\gamma_2 dV_2$, $\gamma_3 dV_3$, etc. (pounds), and if we substitute these for the forces P_1, P_2, P_3, etc., in the expression for the $\bar{x}$ of the centre of parallel forces (foot of p. 16, M. of E.), we obtain

$$\bar{x} = \frac{x_1\gamma_1 dV_1 + x_2\gamma_2 d_2 V + x_3\gamma_3 dV_3 + \ldots}{\gamma_1 dV_1 + \gamma_2 dV_2 + \gamma_3 dV_3 + \ldots}, \quad . \quad . \quad . \quad \text{(a)}$$

which in the compact notation of calculus, the particles being taken "infinitely small" and, therefore, "infinite" in number, can be written

$$\bar{x} = \frac{\int x\gamma dV}{\int \gamma dV}, \quad \text{or} \quad \bar{x} = \frac{1}{G}\int x\gamma dV, \quad . \quad . \quad . \quad . \quad \text{(b)}$$

where G is the total weight of the body, $= \int \gamma dV$.

If the body is *homogeneous*, all the particles have a common "heaviness," which we may call γ_m and factor out, thus obtaining

$$\bar{x} = \frac{\gamma_m[x_1 dV_1 + x_2 dV_2 + \ . \ . \ .]}{\gamma_m[dV_1 + dV_2 + \ . \ . \ .]}, \quad \text{or} \quad \bar{x} = \frac{\gamma_m \int x dV}{\gamma_m \int dV},$$

from which the γ_m can be cancelled, leaving

$$\bar{x} = \frac{1}{V}\int x dV, \qquad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(c)}$$

where V denotes the total volume of the body. (Note that the factoring out of a common multiplier from a parenthesis corresponds to taking a constant outside of the integral sign.)

109. The Time-velocity Curve and its Use.—From eq. (I.), p. 50, M. of E., we have $ds \div dt = v$, the velocity of a moving point at any instant. Hence, also, $ds = vdt$, or the element of distance equals the product of the velocity at that instant by the element of time. If now, whatever the character of the rectilinear motion, we conceive a curve to be plotted, in which the time (from some initial instant) is laid off as an abscissa, and the velocity of the moving point as an ordinate, this curve may be called a "*time-velocity*" *curve* for the particular kind of motion, being different for different kinds of motion. (Of course, proper scales must be selected in laying off distances on the paper to represent the quantities time and velocity.)

The equation expressing the relation between the two variables, time and velocity, may be regarded as the equation to the curve. Thus, in *uniform* motion we have the velocity v constant, so that the curve is a right line parallel to the horizontal axis, or axis of time, as AL in Fig. 130, where OA represents the constant velocity. If the motion is *uniformly accelerated*, that is, has a constant acceleration, we have $v = v_0 + pt$ [where p is the *acceleration*, or rate of change of the velocity, and v_0 is the initial velocity (for $t = 0$)], and note that the quantity pt, or total gain in velocity over the initial velocity v_0, is directly proportional to the time, so that the curve obtained is a straight line inclined to the axis of abscissas, as BH in Fig. 130, BO representing the initial velocity v_0.

In general, let KVM, Fig. 129, be the time-velocity curve for any rectilinear motion. Here we note that the product $v \,.\, dt$ at any instant during the motion is represented by the area of the vertical strip VR, whose width is dt and length v (by scale). Hence the element of distance ds (feet) is proportional to that area, and the whole distance (s) described from the beginning is represented by the sum of the areas of all such strips from OK to VR, i.e., by the area $OKVR$; while the distance (s_n) described between $t = 0$ and $t = t_n$ will be represented by the complete area $OKVMN$.

If now ON be regarded as the base of the figure $OKMN$, we may conceive of a rectangular figure $OALN$ having the same area and same base as $OKVMN$, and having some altitude AO which can be looked upon as representing the "average velocity" of the motion between $t = 0$ and $t = t_n$. By "average velocity" would be meant that *constant* velocity necessary in a *uniform motion* to enable a moving point to describe the distance s_n in the same time t_n as in the actual motion. In other words, the "average velocity" is the result obtained by dividing the whole distance by the whole time, and is represented by the altitude AO, since the area of a rectangle is equal to the product of its base, $ON (= t_n)$, by its altitude, AO.

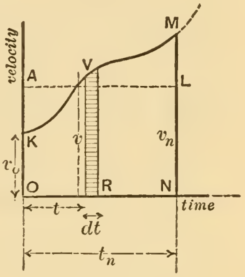


FIG. 129.

As a useful instance consider again a uniformly accelerated motion. Its time-velocity curve is a straight line, BH, Fig. 130, the initial velocity v_0 being represented by OB, and the final, v_n (for $t = t_n$), by HN. The distance described is represented by the area of the trapezoid $OBHN$. This area is equal to that of a rectangle of the same base ON (i.e., t_n) and of an altitude OA = half the sum of OB and NH; i.e., $s_n = \frac{1}{2}(v_0 + v_n)t_n$. In other words, the average velocity between $t = 0$ and $t = t_n$ for the uniformly accelerated motion is $\frac{1}{2}(v_0 + v_n)$.

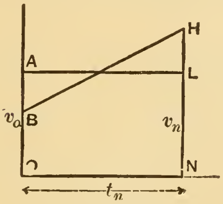


FIG. 130.

If the initial velocity of the uniformly accelerated motion is zero, the time-velocity curve becomes a straight line passing through the origin, O, viz., OH, Fig. 131; and in that case the distance s_n is represented by the area of the triangle OHN, and the average velocity OA is one half the final, or *the final is double the average.* Therefore to obtain the whole distance described we must multiply the whole time by one half the final velocity; i.e., $s_n = \frac{1}{2}v_n t_n$.

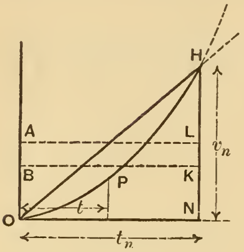


Fig. 131.

If, then, in this case of uniformly accelerated motion with initial velocity = zero, we divide the whole distance by the whole time, *we do not obtain the final velocity* (a common error with students), but only the "average velocity," in the sense defined above. This result must then be doubled to obtain the final velocity.

As an instance where the average velocity is *one third* of the final (the initial velocity being zero), consider the case of variably accelerated motion represented by the relation $v = qt^2$, where q is a constant. The time-velocity curve will have the form OPH, Fig. 131, a parabola with vertex at O. The area $OPHN$ will be one third of that of the circumscribing rectangle, and hence is equal to $\overline{ON} \times$ one third of $\overline{NH}$, i.e., to $\overline{ON} \times \overline{OB}$. Hence OB, or one third of the final velocity, is the average velocity.

Or, mathematically, in detail, $ds, = vdt, = qt^2dt$; therefore

$$s_n = q\int_0^{t_n} t^2dt = \tfrac{1}{3}qt_n^3 = [\tfrac{1}{3}qt_n^2]t_n. \quad . \quad . \quad . \quad (1)$$

But for $t = t_n$ we have $v, = v_n, = qt_n^2$, and hence

$$s_n = (\tfrac{1}{3}v_n)t_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, the average velocity is equal to one third of the final.

110. Reduction-Formulæ for Moment of Inertia of a Plane Figure.—(To replace § 88, M. of E., as regards the *Moment of Inertia* of a plane figure.)

Definition.—Any right line containing the centre of gravity of a plane figure is called a "*gravity axis*" of that figure.

Theorem.—The moment of inertia of a plane figure about a given axis in its own plane is equal to its moment of inertia about a gravity axis parallel to the given axis, augmented by the product of the area of the figure by the square of the distance between the two axes.

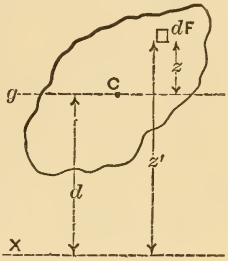


FIG. 132.

Proof.—Fig. 132. Let dF be the area of any element of the plane figure, and z' the distance of that element from any axis X in the plane of the figure; while z is its distance from a "gravity axis," g, *parallel* to axis X. By definition we have $I_x = \int z'^2 dF$; but $z' = z + d$, d being the distance between the two axes. Hence

$$I_x = \int (z + d)^2 dF = \int z^2 dF + 2d \int z dF + d^2 \int dF.$$

Now, from the theory of the centre of gravity, we have (see eq. (4), p. 19, M. of E.) $\int z dF = F\bar{z}$, where $\bar{z}$ is the distance of the centre of gravity C from the axis g. But g is a gravity axis, so that $z = 0$; and hence $\int z dF = 0$. Also $\int dF = F$, the whole area; and $\int z^2 dF = I_g$. Whence, finally,

$$I_x = I_g + Fd^2. \quad . \quad . \quad . \quad (4) \quad \text{Q. E. D.}$$

It also follows, by transposition, that

$$I_g = I_x - Fd^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

which shows that the moment of inertia of a plane figure about a gravity axis is smaller than that about any other axis parallel to that gravity axis.

The moment of inertia of a plane figure plays an important part in the theory of beams subjected to bending action, the transverse section of the beam forming the plane figure in question; somewhat as the mere area of the section does when the beam or rod is subjected to a straight pull.

111. Miscellaneous Examples. (See opposite pages for figures.)

1. Fig. A. Given the data of the figure, find the stress in every two-force piece, and the three pressures exerted on the pin at *A*.

2. Wind from the S.W., 30 miles per hour. Ship going toward the N.W. at 10 miles per hour. At what angle with ship's course should a vertical sail be placed that the air-particles may strike it at an angle of 30° on the hinder side?

3. Locate, by calculus, the centre of gravity of the plane figure in Fig. B, the equation to the upper bounding curve being $xy = 20$ sq. ft.

4. In the rectilinear motion of a material point weighing 12 lbs., and moving horizontally on a rough surface, friction from which is the only horizontal force, we note that positions *A*, *B*, *C*, and *D* are passed at the following times (by the clock), respectively:

3 h. 4 m. 8.1239 sec.; 3 h. 4 m. 8.2350 sec.; 3 h. 4 m. 8.3490 sec.; 3 h. 4 m. 8.4658 sec.

The distance from *A* to *B* is 10.00 ft.; from *B* to *C*, 10.02 ft.; and *C* to *D*, 10.05 ft.

Find (approx'ly) the acceleration of the motion as the material point passes the position *B*; also for *C*; and what must be the value, in pounds, of the friction at *B*?

5. Fig. C. Find the stress in each two-force piece, and pressure on pin at *A*; also pressure of pin *B* on bar *BD*.

6. Of the solid right cylinder in Fig. D, the internal conical portion is of lead, whose spec. grav. is 11.3, while the remainder is of cast iron. Find the centre of gravity of the whole solid.

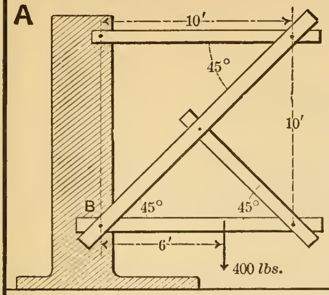
7. In Fig. 80 on p. 67, Notes, compute the pressure between the block and the guide when the former, having started from rest at *A*, is passing position *B*, 45° from *A*.

8. Fig. E. The block weighs 40 lbs. The cord *d* . . . *e* is attached to it at angle 15° with the plane, which is smooth. If the block is to be permitted to slide from rest down the plane, under action of gravity, the cord, and the plane, what constant tension (lbs.) must be maintained in the cord that a velocity of 12 ft. per sec. may be generated in 2 seconds? Afterwards, what new value must be given to this tension if the velocity is to continue at that figure (12 ft. per sec.)?

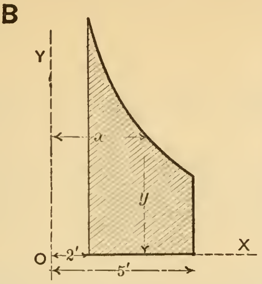
9. In the vertical fall of a material point, certain consecutive small space-intervals are given, $= a$, b, c, and d; and the corresponding time-intervals, t, t', t'', t'''. Derive approximate formulæ for the velocities at mid-points of these time-intervals; the accelerations near the end of some of these intervals; and the corresponding resultant force that must be acting, the weight of the body being G.

10. Fig. F. Find the amount and position of the pressure between the bar *AB* and each of the four pins passing through it; also the stress in *DB*.

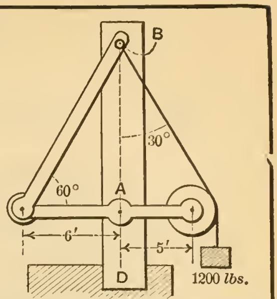
A



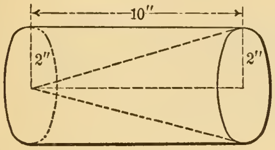
B



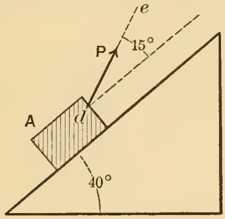
C



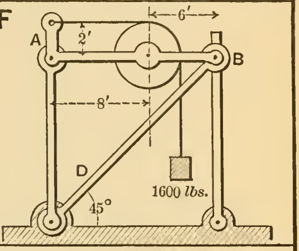
D



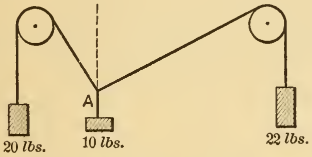
E



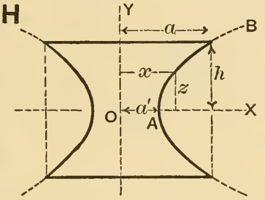
F



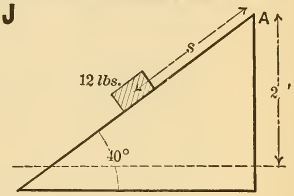
G



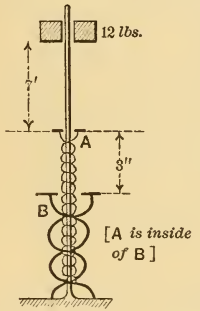
H



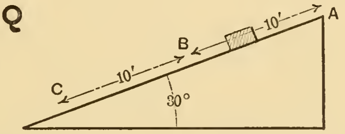
J



R



Q



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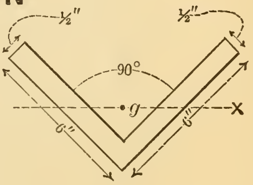
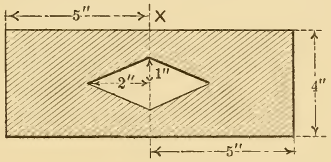
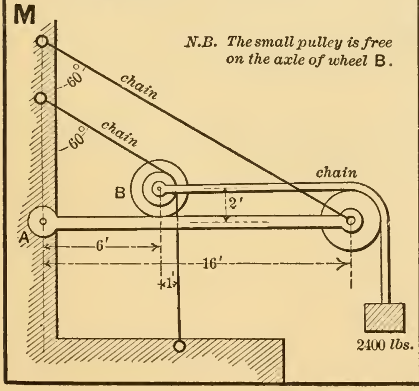


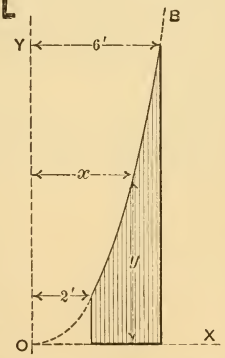
K



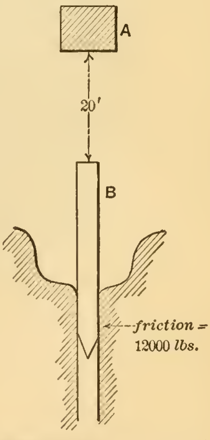
M



L



P



11. Fig. G. The two pulleys run on fixed bearings, without friction. If the given three weights be allowed to come to a position of equilibrium, the knot A being a *fixed knot*, find the angles the two oblique cords make with the vertical, respectively.

12. Find the moment of inertia about the axis X of the symmetrical plane figure shown in Fig. H. The equation to the bounding curve AB is $(x - a') \div (a - a') = z^2 \div h^2$.

13. Fig. J. The block weighs 12 lbs., and in sliding down the rough inclined plane encounters a variable friction, which in lbs. = 3 times the distance, s, from the starting point, in feet. It starts from rest at A. Find the velocity acquired on its reaching a position 2 ft. vertically below A.

14. Compute the moment of inertia of the plane figure in Fig. K about the axis X; also the corresponding radius of gyration.

15. Compute the y co-ord. of the centre of gravity of the plane figure shown in Fig. L. The equation to bounding curve is $x^3 = 4y$, with the foot as linear unit.

16. A block of 50 lbs. weight is started along a rough horizontal table with an (initial) velocity of 40 ft. per sec. If the friction met with is *variable*, and, in lbs., equal to 700 times the body's weight $\div$ (1200 + the square of the veloc. in ft. per sec.), find the time and distance in which the body comes to rest.

17. Fig. M. Find the tensions in all chains and the pressure on pin A and under wheel B.

18. Fig. N. Find the moment of inertia of the plane figure about the gravity-axis X which is perpendicular to the axis of symmetry.

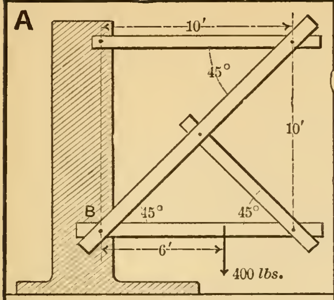
19. Fig. P. The ram A, of 1200 lbs. weight, falls from rest through 20 ft., and has then an *inelastic* impact with the pile B, of weight 400 lbs. Compute their common velocity after the impact, the Kinetic Energy lost in the impact, and the distance the two bodies will sink after the impact, overcoming the constant frictional resistance of 12000 lbs. on side of hole.

20. Fig. Q. Block of 10 lbs. weight on inclined plane. It starts from rest at A. If friction on AB is 2 lbs., while on BC it is 3 lbs., compute the time of reaching position C, and its velocity at that instant.

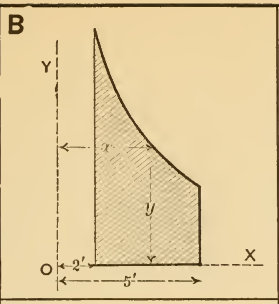
21. Fig. R. A block of 12 lbs. weight falls from rest, freely through the first 7 ft., but then strikes the head of spring A, which opposes a resisting force at rate of 100 lbs. per inch of shortening; and 3 inches further down strikes spring B, offering 160 lbs. of resistance per inch of shortening. Where is the block when (momentarily) brought to rest (supposing the elastic limit of the springs not passed)?

22. A body weighing 12 lbs. is given an initial upward vertical velocity of 10 ft. per sec., being thereafter acted on only by gravity and a variable horizontal force = [$\frac{1}{10}$ of time in sec.] lbs. Find the equation to its path.

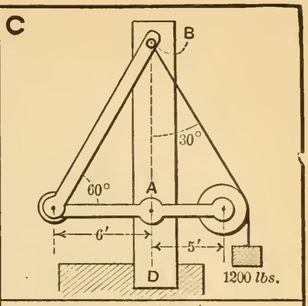
A



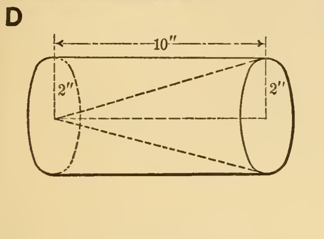
B



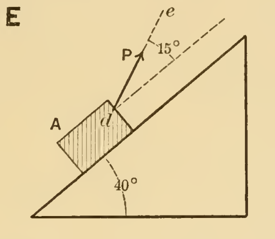
C



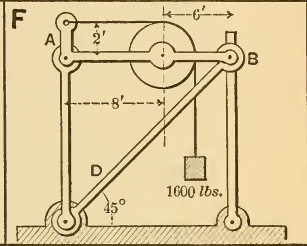
D



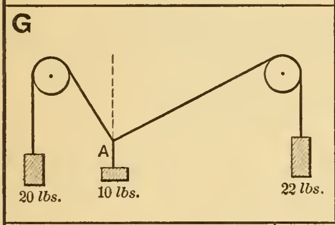
E



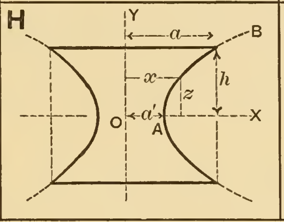
F



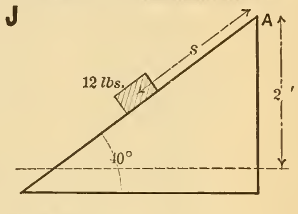
G



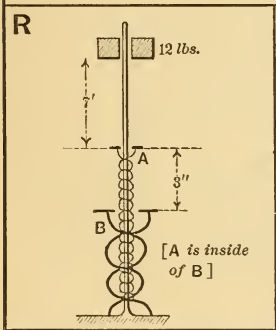
H



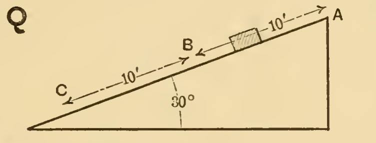
J



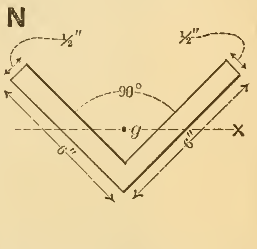
R



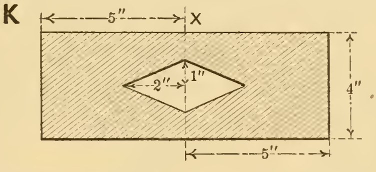
Q



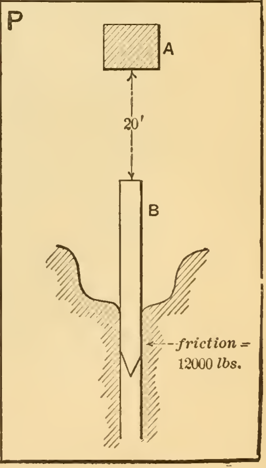
N



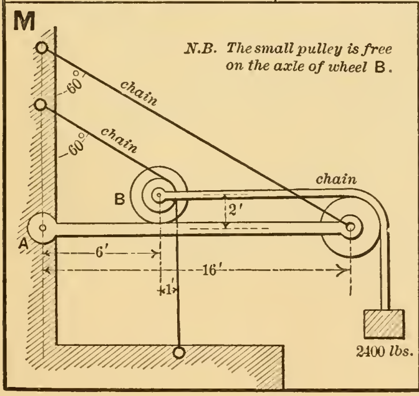
K



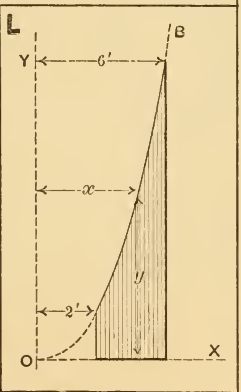
P



M



L



112. Answers to Preceding Problems.—1. 339.1 lbs. and 240 lbs.; on pin A, 240 lbs. vertically, 288.3 lbs. at 33° 42′ with horizontal, and 466.2 lbs. at 30° 56′ with vertical.

2. 41° 34′; or sail pointing 3° 26′ N. of West.

3. $x = 3.28$ ft.; $\bar{y} = 3.28$ ft.; area of figure = 18.31 sq. ft.

4. In passing B, acc. = 18.72 ft. per sec. per sec.; C, 16.05; friction as the material point passes B, 6.98 lbs.

5. 154 lbs. compression, and 1200 lbs. tension; at A, pressure = 738 lbs. at 25° with horiz.; pressure of pin B on bar BD = 1132 lbs. at 36° 45′ with vertical.

6. $\bar{x} = 5.39$ inches from left-hand base.

7. Pressure is 42.44 lbs.

8. First value of tension, 18.92 lbs.; second, 26.63 lbs.

9. Near the end of the first interval the acceleration = $\dfrac{2(bt - at')}{tt'(t + t')}$; that near end of second interval, $\dfrac{2(ct' - bt'')}{t't''(t' + t'')}$.

10. Pressures are 1600, 457, 2263, 1143 lbs., respectively. There is no stress in piece DB.

11. On the left, 87° 43′; on the right, 65° 17′.

12. The moment of inertia, about X, is $\frac{4}{5}[ah^3 + \frac{2}{3}a'h^3]$.

13. The velocity acquired is 7.13 ft. per second.

14. $I_x = 330.66$ biquad. inches; rad. gyr. = 3.02 inches.

15. The area is 80 sq. ft. and $\bar{y} = 15.61$ ft.

16. Time = 3.075 sec., to come to rest. Distance = 70.9 ft.

17. Tension in upper chain, 5838 lbs.; in the lower oblique chain, 2770 lbs. Pressure under wheel B, 1385 lbs. Pressure on pin at A is 7503 lbs., and is directed toward the right at an angle of 6° 37′ above the horizontal.

18. Distance of centre of gravity from the horizontal line drawn through the upper corners is 2.20 inches. Moment of inertia about the gravity axis X is 8.12 biquad. inches.

19. 26.91 ft. per sec.; 6000 ft.-lbs. lost; 1.73 ft. sinking after impact.

20. Time, A to C = 2.06 sec.; velocity at C = 18.05 ft. per sec.

21. The block is 7 ft. 4.297 inches below its starting-point, i.e., has shortened spring A an amount of 4.297 inches.

113. Examples.—1. Fig. 149. The hollow shaft A is to be *twice as strong* torsionally (i.e., as to torsional moment) as the solid shaft B; while the *material* of A is only half as strong as that of B. Given the lengths l and l'', the radius r'' of B, and the outer radius, r, of A, determine a proper value for r', the inner radius of A, for above conditions.

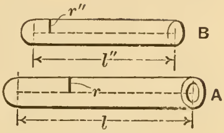


FIG. 149.

2. Fig. 150. Homogeneous prismatic beam; rectangular section; width $= b$ and height $= h$; placed with b horizontal, and loaded uniformly over its whole length at rate of w lbs. per running inch. Given the *whole* length, l, and position of support A (at left *end*), *where* (i.e., distance $a = ?$) shall we place support B that the moment of stress-couple in section over B shall equal (without regard to sign) the greatest moment of stress-couple occurring between A and the point of inflection, F, of the elastic curve?

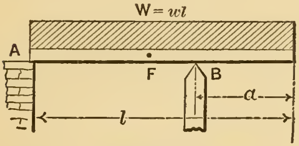


FIG. 150.

Also, after a is found, find the maximum shear. (b and h are given, and also w; elastic limit supposed not passed.)

3. Fig. 151. By the principles of the *graphical statics of mechanism*, having the resisting force Q given in amount and position, given also all friction angles concerned, and considering *all* kinds of frictional action, *find* the value of P for forward motion; also for backward motion; the efficiency of the mechanism.

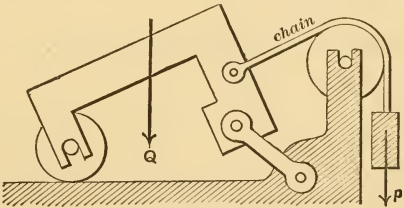


FIG. 151.

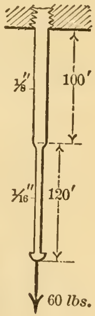


FIG. 152.

4. Fig. 152. The load, of weight $P = 60$ lbs., is gradually applied at lower end of compound vertical round wire (of wrought iron) whose upper end is fixed. Neglecting the weight of the wire, compute the total elongation of the wire and the *work done* on the wire

during the gradual stretching. (Lengths are 100′ and 120′; diameters $\frac{1}{8}$ in. and $\frac{1}{16}$ in., respectively.)

5. Fig. 153. The horizontal prismatic beam is of rectangular section; width b, horizontal. Its height h is to be *three times* its width. Beam of *timber.* The length and character and amount of loading are given in the figure, the weight of the beam itself being neglected.

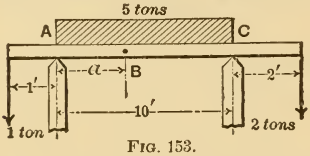


FIG. 153.

There being *three* (local) maximum moments (of stress-couple), viz., at A, B, and C, locate section B by determining distance "a"; compute the moments at A, B, and C; and then determine the *minimum safe* dimensions to be given to the section of the beam. (The load of five tons is uniformly distributed along the ten feet.)

6. Fig. 154. The short vertical cylindrical body *abcd* is fixed at the upper end while sustaining at its lower end a weight of 4000 lbs., whose line of action prolonged upward passes through the extreme edge, a, of the section ab.

The horizontal section of the body is a *circle* of radius = 0.5 inch. It is required to compute the stress per square inch at point a of the section ab, and also that at point b.

Also, what would these stresses be if the line of action of the 4000 lbs. load passed through the *centre* of the section ab?

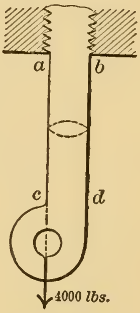


FIG. 154.

114. Answers to Problems in § 113.

1. $r' = \sqrt[4]{r^4 - 4rr''^3}$.

2. $a = (1 - \sqrt{\tfrac{1}{2}})l, = 0.293l$. The maximum shear is just on the left of support B and is $J_m = (\sqrt{2} - 1)wl = 0.414wl$.

3. (Solved graphically.)

4. Total elongation = 1.21 inches. Work done = 36.3 in.-lbs.

5. Distance a = 4.4 ft. Moments are 1.00, 3.84, and 4.00

ft.-tons, respectively. Taking $R' = 1000$ lbs. per sq. in., we have $h = 12$, and $b = 4$, in.

115. The "Imaginary System."—In conceiving of the imaginary equivalent system in § 108, M. of E., applied to the material points supposed destitute of mutual action, and not exposed to gravitation, we employ the simplest system of forces that is capable, by the Mechanics of a Material Point, of producing the motion which the particles actually have. If now the mutual actions, coherence, etc., were suddenly re-established, there would evidently be no change in the motion of the assemblage of particles; that is, in what is now a rigid body again; hence the imaginary system is equivalent to the actual system.

In applying this logic to the motion of translation of a rigid body (see § 109 and Fig. 122, M. of E.) we reason as follows:

If the particles or elementary masses did not cohere together, being altogether without mutual action and not subjected to gravitation, their actual rectilinear motion in parallel lines, each having at a given instant the same velocity and also the *same acceleration*, p, as any other, could be maintained only by the application, to *each* particle, of a force having a value = *its mass* × p, directed in the line of motion. In this way system (II.) is conceived to be formed and is evidently composed of parallel forces all pointing one way, whose resultant must be equal to their sum, viz. $\int dM \times p$. But since at this instant p is common to the motion of all the particles, this sum can be written $p\int dM$, = the whole mass $M \times p$.

If now the mutual coherence of contiguous particles were suddenly to be restored, system (II.) still acting, the motion of the assemblage of particles *would not be affected* (precisely as the falling motion *in vacuo* of two wooden blocks in contact is just the same whether they are glued together or not) and consequently we argue that the imaginary system (II.), is the equivalent of whatever system of forces the body is actually subjected to, viz. system (I.), (in which the body's own weight belongs) producing the actual motion.

Since the resultant of system (II.) is a single force, $= Mp$, parallel to the direction of the acceleration, it follows that the resultant of the actual system is the same.

116. Angular Quantities—Rotary Motion about a Fixed Axis.—In Fig. 155 let the body MLN rotate about the fixed axis C (perpendicular to paper), the initial position of the arm CL having been CY. Now the angular motion of the whole rigid body is the same as that of the arm CL. If in a small time-interval, dt, the arm passes from position CA to position CB, and thus describes the small angle ACB, whose value in π-measure is $d\alpha$ radians, the angular velocity at about the mid-point of angle ACB is $\omega = d\alpha \div dt$. In the next and equal time-interval a slightly different angle, $d\alpha'$ radians, is described; and if in the figure we lay off angle BCE equal to ACB, the angle ECB, or difference between $d\alpha'$ and $d\alpha$ may be called $d^2\alpha$. And so on, for any number of consecutive $(d\alpha)$'s, described in equal times, each $= dt$. If the motion is *uniform*, all the $(d\alpha)$'s are equal; that is, the angular velocity is constant, each $d^2\alpha$ being $=$ zero.

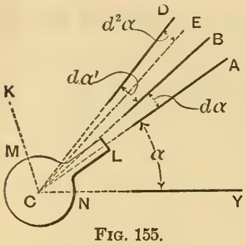


FIG. 155.

If all the $(d^2\alpha)$'s are equal, the motion is *uniformly accelerated*, the angular acceleration, θ, being thus determined: The gain, or change, of angular velocity occurring between the mid-point of ACB and that of BCD is $\frac{d\alpha'}{dt} - \frac{d\alpha}{dt}$, or $\omega' - \omega$; and if this be divided by the time, dt, occupied in acquiring the gain, we have for the *rate of change of angular velocity*, that is, for *the angular acceleration*, the value $\theta = \frac{\omega' - \omega}{dt} = \frac{d\omega}{dt}$.

Another form is this: if the gain of angular velocity, $\frac{d\alpha'}{dt} - \frac{d\alpha}{dt}$, be divided by dt, we have

$$\theta = \frac{d\alpha' - d\alpha}{(dt)^2} = \frac{d^2\alpha}{dt^2}. \quad . \quad . \quad . \quad . \quad \text{(VII)}$$

If the successive $(d^2\alpha)$'s are unequal (the successive $(d\alpha)$'s being described in *equal* time-intervals, it must be remembered), the motion is some kind of *variably* accelerated angular motion; e.g., in a harmonic (oscillatory) rotary motion the angular acceleration at any instant is directly proportional to the angular space between the "arm" or reference line of the body and the middle of its oscillation, and is of contrary sign; i.e., $\theta = -A\alpha$, where A is a constant.

Example 1. At a certain part of its revolution a fly-wheel is found to describe just 1° in 0.01 second. Here we have $d\alpha = 0.01745$ radian, and dividing this by the $dt, = 0.01$ sec., we obtain $\omega = 1.745$ radians per second as the angular velocity of the wheel at this part of its progress, as nearly as the data permit. (Strictly, this value of ω is only the average value of the angular velocity for this small but finite portion of the motion. The data are insufficient to determine whether the velocity is variable or constant.)

Example 2. The same wheel, besides describing 1° in 0.010 sec., is found to describe 1° 2′ in the next 0.01 second. Compute the angular acceleration, as near as may be from these data, for this part of the motion. As before, $d\alpha = 0.01745$ radian, and we now have the additional fact that $d\alpha' = 0.01803$ radian, each of the time-intervals being $dt = 0.010$ sec. If we substitute directly in eq. (VII), there results

$$\theta = \frac{0.000581 \text{ radian}}{(0.01 \text{ sec.})^2} = 5.81 \text{ rad. per sec. per sec.}$$

That is to say, at this part of the wheel's motion its angular velocity is increasing at the rate of 5.81 velocity-units per second, i.e., 5.81 radians per second per second.

Another method is this: The velocity at the mid-point of the $d\alpha$ is, as before, $0.01745 \div 0.01, = 1.74$ radians per second, while at the middle of the $d\alpha'$ it is $0.01803 \div 0.01, = 1.803$ radians per sec.; taking the difference of which we find that 0.058 rad. per sec. of angular velocity has been gained while the wheel was passing between these two mid-points. Hence, dividing this gain by the time of passage, 0.01 sec., we obtain $\theta = 0.058 \div 0.01 = 5.81$ rad. per sec. per sec.

APPENDIX.

NOTES

ON THE

GRAPHICAL STATICS OF MECHANISM.

PREPARED BY I. P. CHURCH, CORNELL UNIVERSITY.

These notes are based mainly on the work of Prof. Herrmann of Aix-la-Chapelle, the earlier form of which was presented as an appendix to Vol. III, of Weisbach's Mechanics.

It is thought that the form of presentation adopted in the following pages is that best adapted for students already familiar with the Graphical Statics of quiescent structures, which Prof. Herrmann assumes is not the case with the readers of his book.

For greater clearness, all force polygons have been placed on separate parts of the paper from the mechanism itself, instead of being superposed on the latter as in Prof. Herrmann's book.

The figures referred to in the text will be found in the back of this pamphlet; while the paragraphs (§) referred to (of a higher number than "30") will be found in the writer's "*Mechanics of Engineering.*"

For table of contents see p. 28.

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1. Assumptions. The forces acting on each part of any mechanism here treated will be considered to be in the same plane and *to form a balanced system,* i.e., to be in equilibrium; in other words, the motions of the pieces take place without sensible acceleration (or the effect of inertia is disregarded). (See p. 440 of Prof. Unwin's *Machine Design* for consideration of the inertia of a piston.) Also, the weights of the pieces will be neglected unless specially mentioned.

2. Graphical Treatment. A "two-force" piece, or "two-force" member of a mechanism is one acted on by only two forces; which forces, therefore, must be equal and opposite and have a

common line of action, for equilibrium. No force polygon need be drawn for such a piece.

A Three-Force Piece. Here, for equilibrium, the three lines of action must meet in a common point and the force polygon is a triangle (see § 325); for example, see Fig. A (bell-crank), (p. 28).

For the equilibrium of a *four-force piece* (Fig. B), the resultant of any two of the forces must be equal and opposite to that of the other two forces. The common line of action of these resultants is the line joining the intersections a and b, while their common amount is given by the ray OC, which is parallel to $a\,..\,b$ and is a diagonal of the closed force polygon (here a quadrilateral). If the four forces act in parallel lines, and two are given, we determine the other two by an "equilibrium polygon," etc., by § 329; i.e., we treat the two unknown forces as pier-reactions, even if their action-lines (one or both) are between the action-lines of the known forces. (For example, see [A] in Fig. 17, Plate V, where from the known forces S_1 and S_2 we construct the two unknown, P and R, in given action-lines, to balance them. The equilibrium polygon begins at c in the left-hand abutment-vertical and consists of segments $c \dots d$, $d \dots e$, and $e \dots f$, terminating in the right-hand abutment-vertical, at f. We then draw $c \dots f$ as the abutment-line, or "closing line.")

No more than four forces will act on any piece.

3. Efficiency. In each of the following problems there will be but one working force or driving force, P, and but one useful resistance, Q; all other resistances being due to friction. For present purposes we are to understand by *efficiency* the ratio of the value P_0, which would be sufficient for the driving force if there were no friction of any kind, in order to overcome Q (without acceleration), to the value P, which it must actually have, to overcome Q under actual conditions, i.e., with friction. Hence efficiency $= \eta = \frac{P_0}{P}$. [Strictly, the efficiency involves the distances traversed; see § 5.]

4. Backward Motion. If the useful resistance Q is a load due to gravity, i.e., a weight, the value to which the working force

(now such no longer) must be diminished in order to allow the mechanism to run backward without acceleration will be called P'; and if in any case P' is found to be *negative*, we recognize the mechanism to be *self-locking;* that is, it will not run backward (or "*overhaul*") when the working force is zero, but, on the contrary, a force must be applied in the action-line of the working force in the opposite direction to cause a backward motion. For instance, in Fig. C, the force P is necessary for the uniform downward motion of the handle A, to raise Q, and overcome friction at all points. With no friction, P_0 would be sufficient to raise Q, and the efficiency $= \frac{P_0}{P}$; whereas, to enable Q to sink uniformly, a still smaller force, viz. P', must be applied at A (but still positive in this case, so that the machine is not self-locking).

5. **Efficiency in General...**, with one working force P and one useful resistance Q, is the ratio of the work Qs' to the work Ps, where s' and s are the respective distances worked through in forward motion by Q and P (simultaneously); i.e.,

$$\eta,\ \text{or } \textit{efficiency},\ = \frac{Qs'}{Ps}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now by § 142, if $\Sigma(Fs'')$ denotes the sum of the amounts of work lost in friction at various points where rubbing occurs in the mechanism, we have

$$Ps = Qs' + \Sigma(Fs''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But in the ideal case of *no friction* (or perfect efficiency), taking the same range of motion as before, and letting P_0 denote the new (and smaller) value of the working force which is now sufficient to overcome Q, we have

$$P_0 s = Qs' + 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Hence from (a), (b), and (1) it is plain that the efficiency may be written thus:

$$\eta = \frac{Qs'}{Ps} = \frac{P_0 s}{Ps} = \frac{P_0}{P}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

the form proposed in § 3 above.

6. Condition of being Self-Locking. If the efficiency is less than 0.50 and the lost work of friction be assumed to be the same in forward as in backward motion (in *same range* of motion, of course), the mechanism is self-locking.

Proof. For forward motion (no acceleration), letting $\Sigma(Fs'')$ denote the sum of the amounts of work lost in friction at the various points of rubbing, we have, for a definite range of motion, (see § 142,)

$$Ps = Qs' + \Sigma(Fs''), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for backward motion (same range), similarly, Q being a a working force and P' a resistance, (see § 142,)

$$Qs' = P's + \Sigma(Fs''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As implied in the notation, assume that the friction-work is the same in backward as in forward motion of the mechanism. From (1) and (2) we deduce

$$\frac{Qs'}{\eta} = Qs' + \Sigma(Fs''); \text{ or, } \Sigma(Fs'') = Qs'\left[\frac{1}{\eta} - 1\right]. \quad . \quad . \quad (4)$$

Substituting from (4) in (3) we obtain, finally,

$$P' = \frac{Qs'}{s}\left[2.00 - \frac{1}{\eta}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (5) it is evident that, when the efficiency is less than 0.50, P' is negative; that is, the mechanism *is self-locking.* The assumption made above as to equality of lost work in forward and backward motion is not exactly true for any machine, perhaps. In most mechanisms the friction work is somewhat less in backward motion, and the proposition is then true if for 0.50 we put a smaller value for η.

Machines of peculiar design *may* be constructed with η greater than 0.50, which nevertheless are self-locking; but in these we find that the lost work is greater in backward than in forward motion.

7. Sliding Friction. By § 156 we know that if two rough surfaces slide on each other the mutual action or force between them is not normal to the plane of contact, but inclined to the

normal at an angle ϕ, the "angle of friction," *on that side of the normal opposed to the direction (of relative motion) of the body under consideration.* Thus, Fig. 1, Plate I, if the block A is sliding toward the right relatively to B (no matter which one, if either, is absolutely at rest) the mutual force between them acts in the line $o \ldots b$; if toward the left, then it acts in the line $o' \ldots b'$. The pressure of B upon A is from m toward o (or o'), that of A upon B from m toward b (or b').

8. Example I. Mill Elevator. (Plate I, Fig. 2.) The single rigid body Ac consists of a platform and vertical side-piece, which rubs at b, the left side of the fixed vertical guide C, and also at a, the right side of the same.

Let Q be the combined weight of the load on the platform and the platform itself, and P the required pull or tension to be applied vertically at A, to maintain a uniform vertical upward motion (*forward* motion, here). Besides the forces P and Q, the body Ac, considered free, is acted on by the forces R_1 and R_2 at the rubbing surfaces a and b, acting at the proper inclination (ϕ) from their respective normals; note which side.

Evidently $A \ldots c$ is a four-force piece. Four action-lines are known, but only one amount, that of Q. To find the amounts of P, R_1, and R_2, we join d (intersection, or "co-point" of P and R_1) with c that of R_2 and Q. On the right of the figure we begin the force-polygon by laying off $m \ldots o$ parallel and equal to Q (by scale). After drawing through m a line parallel to R_2, and through o a line parallel to $c \ldots d$ (the known action-line of the resultant of Q and R_2), by the intersection k we determine $R_2 = \overline{m \ldots k}$, and the resultant, $\overline{o \ldots k}$, of Q and R_2. But this resultant should balance P and R_1, and therefore close a triangle with them in the force-polygon; hence parallels to P and $R_1 \ldots$ through k and o respectively, finally complete the force quadrilateral $m \ldots n$, and fix the values of P and R_1, which can now be scaled off.

In this figure, for clearness, a large value, about 20°, has been given to ϕ, so that $f, = \tan \phi, =$ about 0.36; and from the force-polygon we find that for $Q = 110$ lbs., P is about 140 lbs. If there were *no friction* R_1 and R_2 would be horizontal, and a

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force $P_0 = Q = 110$ lbs. would be sufficient, vertical and upward at A, to maintain a uniform upward motion (or to permit a uniform downward motion, once started). Hence the efficiency for upward (forward) motion, with friction, is $\eta = \frac{P_0}{P} = \frac{110}{140} = 0.78$ or about 78 per cent.

In uniform *backward motion* with friction, R_1 and R_2 (now call them R_1' and R_2') will change their action-lines to the other sides of their normals; i.e., they will now act along $a \,..\, d'$ and $b \,..\, c'$, respectively, at an angle ϕ with the normals. Hence the line $c' \,..\, d'$, in backward motion, takes the place of $c \,..\, d$, in getting the new force-polygon (see on left in Fig. 2); i.e., $o' \,..\, k'$ is drawn parallel to $c' \,..\, d'$, and $m' \,..\, k'$ parallel to $c' \,..\, b$, Q being known. Then a vertical through k' and a parallel to $d' \,..\, a$ through o' complete the polygon; which fixes P' as well as R_1' and R_2'. For $Q = 110$ lbs., P' is roughly about 80 lbs.

Since P' is upward (i.e., not negative, being in the same direction as P) the machine is not self-locking. To alter the design, however, so that it shall be self-locking, we need only place the rubbing point a near enough to b to cause $a \,..\, d'$ to pass through, or below, c'; for then $o' \,..\, k'$ will either coincide with $o' \,..\, n'$ or pass below it, thus making P' either 0 (zero) or negative. But the pressures R_1 and R_2 will be enormously increased. Also, P for upward motion will be larger, and the efficiency smaller than before.

9. **Example II. Wedge.** (Plate I, Fig. 3.) Required the necessary horizontal force P, at the head of the wedge AB, to raise the load Q and overcome the friction at the three points of rubbing, a. c, and e. For the upward motion of the block E (wedge moving to the right) the lines of pressure at these points of rubbing are $a \,..\, b$, $c \,..\, e$, and $c \,..\, d$; there being no pressure at E. The block Ee is a three-force piece, under the action of the *known* Q and the unknown R_2 and R_1 (i.e., the R_1 which acts from c toward d) all three action-lines being given. Hence the force-triangle is immediately drawn, viz., $m \,..\, n \,..\, o$, and the amounts of R_2 and R_1 become known by scale.

The wedge AB is also a three-force piece, acted on by the R_1

(now found) pointing from c toward b, and the unknown R_3 and the required working-force P. Its force triangle $n\,..\,o\,..\,k$ is then drawn, since we have just found R_1 and have only to make $o\,..\,k$ horizontal (i.e., parallel to P) and $n\,..\,k$ parallel to $a\,..\,b$, to close the triangle and thus determine $P = \overline{o\,..\,k}$; (forward motion with friction).

With *no friction*, R_3 is vertical, R_2 horizontal, and R_1 follows the normal to the plane $A\,..\,B$; and the corresponding force-triangles, beginning with the known Q, are $m_0\,..\,n_0\,..\,o_0$ and $n_0\,..\,o_0\,..\,k_0$; and thus P_0 is found.

From the drawing we have, with $Q = 102$ lbs., $P = 98$ lbs. and $P_0 =$ about 30 lbs.; so that the efficiency $\eta = \frac{30}{98} = 0.31$. The angle ϕ used is about 20° or $f = 0.36$.

For *backward motion*, R_1' would act parallel to $c\,..\,d'$, R_2' parallel to $e\,..\,c'$, and R_3' parallel to $a\,..\,b'$; and the corresponding force-triangles are $m'\,..\,n'\,..\,o'$ and $n'\,..\,o'\,..\,k'$, the resulting value of P' being *negative*. That is, P' must act from *right to left*, for backward motion. Hence the mechanism is *self-locking* (for these particular data, in Fig. 3).

Note.—It can easily be shown that if the angle of the wedge (i.e., the angle between its two sides) is less than *twice* the angle of friction (supposed the same for all three rubbing contacts) the mechanism is self-locking. (This is best shown graphically.)

10. Example III. The Jack-Screw. (Plate I. Fig. 4.) Required the value of each force P of a couple, in a horizontal plane, applied to the cross-bar $a_0\,b_0$, in order to raise the load Q and overcome the friction between the surfaces of the square-threaded screw and nut. The screw-shaft is vertical. Assume no pressure at the edges of the threads. The pressures on the helical surfaces may be considered concentrated at two points d_1 and d_2, diametrically opposite, in the middle of the width of the thread. The pressure at d_1, viz. R_1, will lie in a vertical plane $\urcorner$ to $a_0\,..\,b_0$, and make an angle $\phi + \alpha$ (on the left) with the vertical, while R_2 on the other side makes an equal angle with the vertical (but on the

right). α is the angle which the helix of screw surface makes with the horizontal.

Projecting the five forces on a plane ⏋ to the bar $a_0\,..\,b_0$, we find that Q must balance the vertical components of R_1 and R_2, which justifies the force-triangle drawn at [B], where, with Q given, we easily fix R_1 and R_2 by drawing $w\,..\,m$ and $k\,..\,m$ at an angle of $\phi + \alpha$ with Q, as shown. Therefore the horizontal component of R_1 is $\overline{p\,..\,m}$, that of R_2 is $\overline{m\,..\,p}$.

Projecting all the five forces on a horizontal plane, we note that the two unknown P's must balance the couple formed by the horizontal components of R_1 and R_2, as if these components (which we will now call R_3 and R_4) were applied directly to the bar $a_0\,..\,b_0$ at points vertically above d_1 and d_2. Hence (see [C] in Fig. 4) we treat the known R_3 and R_4 (each $= \overline{m\,..\,p}$) as parallel forces applied perpendicularly to a straight beam $a\,..\,b$ supported at a and b on || smooth surfaces whose normals are ⏋ to $a\,..\,b$ and || to R_3 and R_4. The reactions of these supports will be || to R_3 and R_4 and are the forces P and P required. At [C] in Fig. 4, $\overline{d_1\,..\,d_2} =$ the distance $\overline{d_1\,..\,d_2}$ of [A], and $\overline{a\,..\,b}$ = distance $a_0\,..\,b_0$.

Hence, using the construction of § 329, we make $\overline{x\,..\,y} = R_3$, $\overline{y\,..\,x} = R_4$, select a pole O at convenience, draw the three *rays* $O\,..\,x$, $O\,..\,y$, and $O\,..\,x$, and a corresponding equilibrium polygon $u\,..\,t\,..\,s\,..\,r$, beginning at any point u in the action-line $u\,..\,b$ of the lower P. Join $u\,..\,r$, the abutment-line, and draw a line || to it through the pole O to fix n'; then $x\,..\,n' = P$ required.

With *no friction*, $\phi = 0$, and R_3 and R_4 are each equal to $\overline{m_0\,..\,p}$, instead of $\overline{m\,..\,p}$; and P_0 is proportionally smaller than P. In this figure P_0 is about two thirds of P; i.e., the efficiency is about 0.66.

For *backward motion*, R_1 and R_2 must act on the other sides of their respective normals; i.e., in [B] we should use $\alpha - \phi$ instead of $\alpha + \phi$, and if in that case α were less than ϕ, the point m would fall on the right of p, and P' would be negative; i.e., the screw would not run backward (would not "overhaul") when there was no force on the cross-bar; (self-locking).

11. Pivot Friction. (Plate II, Fig. 5.) Since the frictions at the base of a flat-ended pivot (see § 168) are equivalent to a couple in which each force is $\frac{1}{2}fR$, or $\frac{1}{2}fQ$ with present notation, and whose arm is $\frac{4}{3}$ of the radius r (so that the moment of the couple is $\frac{2}{3}fQr$), we may suppose the pressure concentrated at two opposite points, in the circumference having a radius $= \frac{2}{3}r$, as in Fig. 5. The pressures at these points are inclined at an angle $= \phi$ with their respective normals and in the proper directions, as shown in Fig. 5. If uniform motion is to be maintained by a horizontal couple of a moment $= Pa$, P being unknown and Q and ϕ given, we may proceed graphically as in the preceding problem, making $\alpha = 0$ (zero); a here denotes the distance $a \, . \, . \, b$ of preceding problem.

12. Journal Friction. Assuming that the journal is fitted to its bearing with sufficient play to permit the line of contact to take *any position* (in the circumference of the bearing) called for by the conditions of equilibrium (i.e., there is no initial clamping), we know from § 163 that the action-line of the mutual pressure between them must be tangent to the friction-circle, when rubbing is taking place. As to which side of the friction-circle is to have the action-line drawn tangent to it, that is a matter depending on the direction of the motion and must be decided by the statement in § 7. The bearing in which the journal turns may itself be in motion (the crank end of a connecting-rod, for instance) and the direction of relative turning must be considered. The following examples will bring out clearly all the relations in such cases. Each friction-circle will be much exaggerated, for clearness, in the figures, and must not be mistaken for a journal. The radius of a friction circle is $r \sin \phi$, where r is the radius of the journal. (Note carefully the relations in § 163.)

13. Example IV. Bell-crank. (Plate II, Fig. 6.) The load Q is given in amount and direction, being supported by a vertical link which is jointed at A to the bell-crank. The working force, P, has a given direction and is applied through a link jointed at B to the bell-crank. Both joints consist of journals in bearings. $P = ?$ to overcome friction at A, B, and C, and to raise Q. The motions at A and B are such that the action-line of Q must be

tangent to A's friction-circle on *its left;* and that of P tangent to B's friction-circle on its *under side.*

Drawing, then, these tangents ‖ to the respective known directions of P and Q, we have $A \,..\, a$ and $c \,..\, a$ as their action-lines, meeting at a, through which (since the bell-crank is a three-force piece) the third force R must pass (the reaction at C), and this must be tangent to the friction-circle at C on its *upper* side. Hence a line through a and tangent to the friction-circle at C on upper side, is the action line of R, viz., $a \,..\, b$. The force-triangle $k \,..\, m \,..\, n$ is now easily drawn, Q being the known force and laid off first, and $P = \overline{m \,..\, n}$ is thus determined. With *no friction*, P, and Q (directions unchanged) must pass through the centres of the pins at A and B, and intersect at a_0. R then passes through a_0 and the centre of C, i.e., acts along a_0b_0. $P_0 = \overline{mn_0}$ in the new force-triangle, and the efficiency $= \overline{m \,..\, n_0} \div \overline{m \,..\, n}$. For *backward motion* we make the action-lines tangent to their respective friction-circles on the *other side* in each case; then $P' = \overline{m \,..\, n'}$.

14. Example V. The Slider-crank. (Plate II, Fig. 7.) The wheel W and the crank B form a single rigid body turning on a fixed bearing C. The connecting-rod or link, BD, is pivoted about the crank-pin at one end and to the cross-head pin at the other. The cross-head block slides in a right line between guides, and receives the pull (or push) of the piston-rod in the axial line of that rod, i.e., through the centre of the pin at D. The resistance Q being given, applied in line $a \,..\, b$ to wheel W, required the necessary value of P for uniform motion *in the given position of the mechanism.* The connecting-rod is a two-force piece, and in its present position (B on the right of a vertical through C) the pressure at the cross-head pin is tangent to the friction-circle there on its upper side; that at the crank-pin on the lower side of the friction-circle there. Hence a line drawn so as to be tangent to the two circles in the manner stated is the action-line of R_1, the crank-pin pressure (as also that at the cross-head pin); i.e., draw $e \,..\, h$.

The three-force piece, WCB, is acted on by the known Q, by R_1, and a pressure at the bearing C, viz., R_2, which is tangent to

the friction-circle at C on the upper side; therefore R_2 must act in the line $b\,..\,f$ drawn from the intersection b, of Q and R_1 tangent to friction-circle at C on upper side. Hence the force-triangle for WCB is easily completed by making $o\,..\,m$ || and $=$ Q, and drawing $m\,..\,n$ and $o\,..\,n$ || to $c\,..\,h$ and $f\,..\,b$, respectively, thus determining R_1 and R_2. Now the cross-head block, Dh, is a three-force piece acted on by R_1 (pointing toward the left), now known; by the unknown P and by the unknown R, which is the pressure coming from the upper guide, and must act in a line $d\,..\,e$ through the intersection of the action-lines of P and R_1, viz., e, and makes angle $= \phi$ (in direction as shown) with the normal to the guide surface. Its force-triangle, then, is rks, $k\,..\,r$ being $=$ and opposite to $\overline{m\,..\,n}$, $s\,..\,r$ being drawn || to P and $k\,..\,s$ || to $d\,..\,e$. Thus P is found, being $= \overline{rs}$.

As for P_0 (i.e., *with no friction*), R_1 would act through the pin centres, thus raising b; R_2 would act through b and the *centre* of C; while R_3 would be normal to the guide surface. With new force-triangles, on this basis, we find $P_0 = \overline{r\,..\,s_0}$; whence the efficiency, $= P_0 \div P$, is found.

In any other position of the mechanism a similar method is available.

15. Example VI. Beam-engine with Evans's straight-line motion. (Plate II, Fig. 8.) WKM is a single rigid body (wheel and shaft) turning on a fixed bearing at K. M is the crank-pin, MB the connecting-rod, DC a link turning in a fixed bearing at D, and pivoted at C to the *beam* ACE, one end of which, E, is guided in a horizontal right line by the block F and straight guides. The pin C being mid-way (in a straight line) between E and A, and the length $\overline{CD}$ being made equal to $\overline{AC}$, which also $= \overline{CE}$ (between centres; D and E at the same level), A will be compelled to move in a vertical straight line, as if it were itself guided by a straight edge. The vertical piston-rod is linked to the beam at A. Required the necessary steam pressure P on the piston-head, for upward motion, the resistance Q being applied to the wheel W in the line $x\,..\,w$, and the parts having the position shown in the figure.

The link DC is a two-force piece, subjected at this instant to

some thrust, R_5, whose action-line must be tangent (below) to the friction-circle at C, and also (above) to that at D. Similarly the link or connecting-rod BM is a two-force piece, under a tension R_1, whose action-line is tangent to the friction-circle both at M and B. At M this tangency is on the right; at B it may be either on the right or the left according as the link has not yet reached, or has passed, a certain critical position which is very nearly the position chosen for the figure; at a venture the tangency has been drawn on the *left* at B.

The block F is a two-force piece, under a thrust R_4, directed, as shown, at an angle ϕ with the normal to the guide-surface below it.

Construction. We first draw the force-polygon for WKM, a three-force piece, the forces being the known Q in line $w..x$, the unknown R_1 in the known line $c..h$, and the unknown reaction R_2 at the bearing K.

R_2 must act in a line $x..y$ through the point x, and tangent, on left, to the friction-circle at K. Hence by laying off $\overline{o..m} =$ and $\|$ to Q, and then drawing $m..n$ $\|$ to $h..c$ and $o..n$ $\|$ to $x..y$, we determine R_1 and R_2.

The beam $ABCE$ is a four-force piece under the forces R_1, R_3, R_4, and R_5, their action-lines being all known and R_1 now known in amount, $= m..n$. The direction of R_3 becomes known from a consideration of the piston and rod which together form a three-force piece, being acted on by P, by the stuffing-box pressure R_6, and by R_3 reversed. Since P and R_6 intersect at a, and since R_3 must pass through a and be tangent on the right to the friction-circle at A, its action-line is easily drawn. Resuming the body $ABCE$, we find the intersection, e, of R_3 and R_1 (N.B. e is off the paper) and join e with f, the intersection of R_4 and R_5. Making $\overline{n_1..m_1} =$ and $\|$ to R_1 (now pointing down), and drawing $n_1..r$ $\|$ to $f..e$ and $m_1..r$ $\|$ to R_3; also $r..u$ $\|$ to R_4 and $n_1..u$ $\|$ to R_5; we finally determine R_3, R_4, and R_5.

Having found R_3, the force-triangle for the three-force piece $A..a$ (piston, etc.) is easily drawn (see [Z] in Fig. 8) and P finally determined.

With *no friction* we find P_0 by taking friction-circle centres

instead of tangencies and making R_6 and R_4 ┐ to their respective sliding surfaces.

16. Example VII. Oscillating Engine. (Plate III, Fig. 9.) Here the connecting-rod is dispensed with, the piston acting directly on the crank-pin A, while the cylinder EE oscillates on trunnions turning in fixed bearings. The crank AB and wheel W constitute a single rigid body turning in a fixed bearing B. The resistance Q being applied to W in line $d..c$, we wish to find the proper steam pressure P on the left of the piston to overcome Q and all intervening frictions, for motion (with insensible acceleration), of the mechanism in the position shown. P acts centrally along the axis of the piston-rod. The pressure R_4 is that of the stuffing-box against the piston-rod, and R_3 that of the cylinder against the edge of the piston. Remembering that the piston and cylinder always have a common axis as they "telescope" in and out, we see that there are only two forces external to these two pieces when considered together, viz., the pressures at the crank-pin and at the trunnion; which two pressures (R_2) must therefore be equal and opposite, both of them acting in the line $h..g$ tangent to the two friction-circles (on the lower side at both A and C). We have thus found the action-line of R_2.

Now consider the three-force piece WBA. The forces are the known Q in line $d..c$, the unknown R_2 in the line $h..g$, and the reaction or pressure R_1 in a line $a..e$ which we draw through e, the intersection of R and Q, and make tangent (on the upper side) to the friction-circle at the bearing B. Hence, making $m..n$ = and || to Q, and $m..r$ and $n..r$ || to $e..h$ and $e..a$, respectively, we fix the amounts of R_1 and R_2 by this force-triangle $m..n..r$.

The piston is a four-force piece, under the action of R_2, now known, acting *toward the left* in line $h..e$; by the unknown P acting along the centre line of the piston-rod; and by the two unknown reactions, R_3 and R_4, inclined at angle ϕ to their respective normals, and acting at known points. Hence join y, the intersection of R_2 and R_3, with x, that of P and R_4; make $o..p$ = and || to R_2, draw $p..s$ || to R_3, and $o..s$ || to $y..x$, to deter-

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mine s; $s..t \parallel$ to P and $o..t \parallel$ to R_4 are then drawn to fix t. P can now be scaled off, as also R_3 and R_4.

With *no friction*, R_2 would act in the axis of the piston through the centres of A and C. In place of e we would have e_0 (not shown in figure), and R would act through e_0 and the centre of B. R_3 and R_4 would be ⏋ to their respective rubbing surfaces. We would then obtain $R_2 = \overline{m..r_0}$ and $P_0 = R_2 = \overline{m..r_0}$; whence the efficiency is $= P_0 \div P = m..r_0 \div s..t$.

16. Example VII. The Blake Ore-crusher. (Plate III. Fig. 10.) K and L are fixed walls, HD oscillates about a fixed bearing D_1, WFa is a wheel and crank rotating on a journal in a fixed bearing F. The connecting-rod $a..c$ communicates motion to the two links or struts, DE and CB, forming a toggle-joint and causing HD_1 to oscillate.

For the given position of the parts, Q being the resistance offered by a piece of ore, h, to any further motion of H toward the left, required the value of P, applied in the line $W..b$ to wheel WF to overcome Q and the pressures at the seven sockets or bearings.

The two links DE and CB are evidently two-force pieces, DE being subjected to some thrust R_2 in line $h..c$, CB to some thrust R_3 in line $c..B$; these lines being drawn tangent to the various friction-circles in the manner shown.

R_2 and Q intersect in h, hence the line $f..h$, drawn through h and tangent to the friction-circle at D_1, must be the action-line of R_1, the reaction at the bearing D_1. That is, Q, R_1, and R_2, are the forces acting on the three-force piece HD_1; therefore, Q and the three action-lines being known, we easily complete the corresponding force-triangle $o..m..n$ and thus determine R_1 and R_2.

Passing to the three-force piece $a..b$ we have the action-lines of R_2 and R_3 already intersecting at c, and the amount of $R_2 = \overline{o..n}$. The third force R_4 has an action-line passing through c and tangent (on left) to the friction-circle at the crank-pin. Hence draw $c..a$ accordingly, and complete the force-triangle by making $\overline{k..i} \parallel$ and $=$ to $\overline{o..n}$, (i.e., to R_2), and $i..r$ and

$k..r$ || to $a..c$ and $B..c$ respectively; which determines R_3 and R_4.

Finally, the three-force piece $a..F..W$ is seen to be acted on by R_4, now known; by the unknown P in line $P..b$; and by the bearing-reaction R acting through b and tangent (on right) to the friction-circle at F. Draw $b..F$, then, as indicated, and the force-triangle is readily formed, $t..s..u$, in an obvious manner, whence $\overline{s..u}$, $= P$, is scaled.

Without friction we would as usual draw the action-lines of the divers R's through the *centres* of the various journals, or sockets, instead of tangent to their friction-circles. With a new set of force-triangles on this basis (see broken lines on the right in Fig. 10), we obtain P_0, considerably smaller than P. In this exaggerated figure the efficiency is only about 0.50; but in Prof. Hermann's drawing, with f' for journal friction $= 0.10$, he obtains a value of 0.80 for the efficiency.

17. **Rolling Friction** (so called). This kind of resistance is due to the fact that the point of application of the force acting between a wheel and the rail or surface on which it rolls is *not* at the foot of the perpendicular dropped from the centre of the wheel upon the rail, but *a little in front* (in direction of rolling) by an amount, or distance, $b =$ about 0.02 in. for iron wheels on an iron rail, and about 0.50 in. for a wagon-wheel on a dry macadamized road ($b =$ from $2.''00$ to $3.''00$ on soft ground). See § 172; (also Prof. Reynolds's article in the Philos. Transac., vol. 166.) As to the *direction* of the pressure of rail on wheel it is somewhere within the cone of friction so long as perfect rolling (i.e., no slipping) proceeds, being the equal and opposite of the resultant of all the other forces acting on the wheel.

Thus, Fig. 11, Plate III, if Q is the weight of the roller, applied in the centre o', the necessary force P, horizontal and acting through the centre of the roller (to maintain a uniform rolling motion), is determined by the fact that the resultant of P and Q must act through c, and therefore along the line $o'..c$, where $\overline{c..d} =$ the distance b just mentioned. Hence, making the line Q through o equal by scale in length and || to Q at o',

and drawing $o \, . \, . \, m \parallel$ to $o' \, . \, . \, c$, as well as a horizontal through the lower extremity of Q, we determine both P and the rail pressure.

Again, Fig. 12, if the rolling action occurs on both sides of the roller, as at c and e when a weighted plank is moved horizontally (to left, here) on loose rollers, we note that it is a two-force piece (neglecting the weight of the roller), the action-line of the compressive forces being $e \, . \, . \, c$, both e and c having been located at distance $= b$ from the perpendicular through the centre (in the proper directions; and then, we may have different b's at the two points of rolling contact). Here the upper plank is moving and the horizontal force (toward left) which must be applied to it to maintain this motion uniformly must be equal to the sum of the horizontal components of the various inclined R's, one from each roller, the sum of the vertical components of the latter being equal to the total load on the plank.

In the uniform motion of a car-wheel, EA, Fig. 13 (brakes not on), the (double) wheel and its axle constitute a single, rigid, two-force piece, acted on by the rail pressure, or reaction, at c ($c \, . \, . \, d$ being made $= b$) and by the pressure of the bearing against the journal or axle at R; and this pressure must be tangent to the friction-circle at E (on the right for motion here shown, car moving to the left). The horizontal component of the equal and opposite of this R is the tractive resistance, on a straight level track with uniform motion (besides the resistance of the air), and is continually overcome by the tension in the draw-bar of the locomotive. R is practically equal to its own vertical component which equals the portion G of the car's weight borne by this wheel, and this horizontal component of R is $= G \tan \theta$, where θ denotes the complement of the angle $e \, . \, . \, c \, . \, . \, d$.

If the brake is in action, Fig. D,* exerting a pressure P' at the point a, this pressure must act along the line $a \, . \, . \, o$ at an angle $= \phi'$ with the radius aE (note the direction of motion, etc.), provided the wheel is not held fast by the brake. Suppose now that the wheel, while continuing to roll without slipping on the rail is *on the point of slipping upon it*, that is, suppose that "skidding" is impending; then P'', the pressure of the rail on the

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* See p. 28.

wheel, besides being applied at c, must take the direction $c..o$, at an angle ϕ'' with the vertical (track level), on the right, here. Since P' and P'' intersect at o, the only remaining force acting on the rigid body EW (which now is a three-force piece) must pass through o. This force is P''', the pressure of the bearing on the journal, and it must also be tangent to the friction-circle at E. Its action-line, therefore, is easily drawn and is $e..o$. A corresponding force-triangle being now constructed (not shown in the figure) with sides ‖ to $c..o$, $a..o$, and $e..o$, respectively, in which the vertical projection of P'' is made $= G$, the portion of weight of car coming on this wheel (double), we determine the value of each force. Since ϕ'' for impending slip (friction of rest) is greater than ϕ' for actual slipping, greater resistance is offered by impending than by actual "skidding." In most cases in practice the distance $c..d$, $= b$, is so small that its effect in graphical work is almost inappreciable.

18. Example VIII. Friction-rollers of Crane. (Plate III, Fig. 14.) At $T..X..Y$ we have a vertical projection of the crane. The horizontal pressure of the crane at the base is exerted through friction-rollers (shown at D and B) against the side of the fixed, conical, and vertical mast $e..b$. The vertical pressure induced by the suspended load being supported at T, while a horizontal pressure simply ($= Q$) is produced at the base $X..Y$, we shall consider the equilibrium of the part $G..E$, carrying the friction-rollers, as if it were acted on by the force Q, applied centrally and symmetrically in the line $a..Q..b$; by the pressures R_1 and R_2 of the friction-roller journals against their bearings; and by a force P, applied on the lug a, and ‖ to face GE. This force P is of such an amount, *to be determined*, as to maintain a uniform motion in direction of the dotted arrow.

That is, $EFGH$ is to be treated as a four-force piece. Now each friction-roller is a two-force piece like the car-wheel in Fig. 13 and the action-line of the compressive forces in each is drawn in the same manner as in Fig. 13, noting well the direction of motion. We thus determine the action-lines, $f..o$ and $d..e$, meeting at o, of the unknown R_1 and R_2.

Note.—Of course the force R_1 with which the bearing of our

four-force piece acts against the journal of the friction-wheel is the equal and opposite of the R_1 with which that journal acts against the bearing. The bearing is part of our four-force piece, and the force R_1 acting upon it is not shown in the figure, but has the same action-line as the R_1 shown.

Having now four action-lines and the *amount* of one force, viz., Q, of the four forces acting on the piece $EFGH$, we proceed to construct the amounts of the other three, in the way so often employed, classifying the set of forces into two pairs. Pair off P and Q; they meet at a; join $a\,..\,o$; lay off $\overline{o''..m}$ = and || to Q; draw $m\,..\,a''$ || to P and $o''\,..\,a''$ || to $o\,..\,a$ thus fixing $P, = \overline{m\,..\,a''}$ Also draw $o''\,..\,n$ || to $d\,..\,e$ and $a''\,..\,n$ || to $o\,..\,f$, thus determining the amounts of R_2 and R_1.

Since, *with no friction*, $P_0 = 0$, the efficiency $= 0$, i.e., all the work done by P is wasted (spent in overcoming friction).

19. Chain Friction. (Plate IV, Fig. 15.) Supposing that the groove in the periphery of the pulley A is properly constructed for receiving a chain in such a way that, as the chain winds upon the pulley, the alternate links place themselves with their planes || to or ⌉ to the axis of the pulley; then, as each link, c, approaches the point K where it is to assume its proper place on the circumference, its neighbor just above being already in place on the pulley, and turning with it (whereas the first one remains vertical for a time) *a turning of one in the hollow of the other*, with corresponding friction, is brought about, and in such a direction that the pressure between the two is tangent to the friction-circle on that side of the latter which is *further* from the centre of the pulley; for one turns in the other like a journal in the bearing. (The pulley in the figure is turning clock-wise.)

Similarly, on the other side, at L, where the chain is unwinding and thus being straightened, each link turns in the hollow of its neighbor, on passing off the pulley, and hence the pressure at L is tangent to the friction-circle there on the side *nearer* to the pulley-axle. Again, at the bearing of the journal of the pulley-axle, the reaction, R, is tangent to the friction-circle and on its right.

Let r' denote the radius of the hollow at the extremity of a

link (equal to the half-thickness of the chain-wire) and r'' the radius of the journal of the pulley-axle; also ϕ' and ϕ'' the respective friction-angles at those points. Then the radius of the friction-circle at L and at $K = a = r' \sin \phi'$, and that at the central bearing $= b = r'' \sin \phi''$. Let r denote the distance between the centre at K (or L) and that at the middle bearing.

We thus see that a weight Q supported on the ascending side of the chain (on the left, here) has a lever-arm $= r + (a + b)$ about the point of application of R; while the vertical force P, applied to the descending (and unwinding) side of the chain, has a lever-arm of only $r - (a + b)$. Hence for uniform motion we have (from equality of movements)

$$P : Q :: r + (a + b) : r - (a + b).$$

From this proportion P may be computed; or, graphically, given Q as a load applied to a horizontal beam with supports at R and P, we may construct R and P as if reactions of those supports, by § 329. Thus, at $[B]$ in Fig. 15 make $\overline{m\,..\,R} \parallel$ and $= Q$; take any pole O and any point e in the vertical action-line of R. Draw the rays $O\,..\,m$, and $O\,..\,R$ and the segments $e\,..\,c$, and $c\,..\,n \parallel$ to them (respectively); join $e\,..\,n$ and make $O\,..\,n' \parallel$ to it through O. Then $\overline{R\,..\,n'} = P$, and $n'\,..\,m = R$.

With *no friction* $P_0 = Q$, and we can find the efficiency.

For *backward motion* the lines of action of the three forces have their friction-circle tangencies each on the side opposite to that in forward motion, and we have

$$P' : Q :: r - (a + b) : r + (a + b).$$

20. Example IX. Pulley Blocks or Tackle. (Plate IV, Fig. 16.) Let C be the upper block of a tackle of two blocks with three pulleys in each (the pulleys turning independently on a common bolt). C is suspended from a support and is stationary, and R is the tension in the rod or shank by which it hangs.

The lower block is movable and carries a load Q, whose uniform upward motion is to be maintained by the application of a proper force P or S_1, to the chain at the extremity b. The chain passes continuously over the six pulleys or sheaves in the manner shown in the figure, the other extremity being attached to the

support above. Each straight part of the chain will be considered vertical, and all these straight parts have different tensions. Denote these tensions by S_1 $(= P)$ S_2, S_3, S_4, S_5, S_6, and S_7.

For simplicity (as will be seen) let us suppose P given and Q required. Since each pulley of this figure is similarly circumstanced to the one in Fig. 15 where the tension on the unwinding side was greater than that on the other in the ratio $\dfrac{r+(a+b)}{r-(a+b)}$ (where a, r, and b have the same meanings as in the preceding paragraph) we see that

$$S_2 : S_1 :: r-(a+b) : r+(a+b);$$
$$S_3 : S_2 :: r-(a+b) : r+(a+b);$$
$$S_4 : S_3 :: r-(a+b) : r+(a+b);$$

and so on, up to

$$S_7 : S_6 :: r-(a+b) : r+(a+b).$$

Hence we adopt the following construction: lay off on a horizontal line, $\overline{o\,..\,w} = r-(a+b)$; $\overline{o\,..\,w'} = r+(a+b)$; and also $\overline{w'\,..\,o'} = r-(a+b)$. Draw a vertical through each of the points o, w, o', and w'. On the vertical through w, lay off to scale $\overline{w\,..\,m} = P = S_1$, and draw and prolong $m\,..\,o'$ to intersect the vertical through w' in some point m'; whence $\overline{w'\,m'} = S_2$. Then a line joining m' with o, by its intersection with the vertical through w, gives a point n such that $\overline{w\,..\,n} = S_3$; while $n\,..\,o'$ prolonged cuts the vertical through w' in some point n', giving $\overline{w'n'} = S_4$; and so on, until all the six tensions S_2 to S_7 are found.

Proof. (By similar triangles we see that the proportions mentioned above are satisfied.)

We then have (see [C] in figure),

$$Q = S_2 + S_3 + S_4 + S_5 + S_6 + S_7; \quad \text{as also}$$
$$R = S_1 + S_2 + S_3 + S_4 + S_5 + S_6.$$

Having thus found the ratio of P to Q, i.e., the value of $\dfrac{P}{Q}$, we can easily compute P if Q is given, as in a practical case; and also the efficiency, since with *no friction* $P_0 = \frac{1}{6}Q$ (for the tackle in this case).

In *backward motion*, having $P = S_1$ given, we would draw a line from m through o, instead of through o', to fix m', and returning, a line through m' and o' to fix n, and so on, each tension so found being greater than the one preceding it. And finally we would have $Q = S_2 + S_3 + S_4 + S_5 + S_6 + S_7$, and the ratio of the given Q to the unknown P' thus fixed; and P' would be found from the relation : $\frac{P'}{Q} = \frac{S_1}{S_2 + S_3 + S_4 + S_5 + S_6 + S_7}$.

21. Example X. The Differential Pulley. (Plate V, Fig. 17.) This is a tackle in which the upper block carries only *one* pulley, which, however, has *two* grooves in || planes, but with slightly different radii. Also, since friction in the grooves is not sufficient for the purpose, projecting pegs or ridges (or some similar device) are provided to *prevent the chain from slipping* in either groove. The lower block D carries an ordinary pulley of one groove, the weight Q being suspended from the axle of this lower block.

The chain is *endless*, passing twice over the pulley A (once in each groove) and once under the lower pulley, while a portion hangs freely, as shown.

Given the load Q, required the vertical force (or load perhaps) P. to be applied to the chain where it unwinds from the *outside* groove (see figure) in order to raise Q and overcome all friction. The lower pulley is then acted on by the vertical force Q, and the two vertical and upward tensions S_1 and S_2 each of these three forces being tangent to its own friction-circle, as shown, on the proper side (note that S_1 is on the unwinding side). We find S_1 and S_2 by treating Q as a load (§ 329) resting on a horizontal beam supported in verticals a and n. At $[B]$ is the force-diagram, while $a\,..\,b\,..\,n$ is the equilibrium polygon. (See § 329.)

S_1 and S_2 having thus been determined, we consider the upper pulley, which is acted on by four || forces, viz., the known S_1 and S_2, the unknown P and the unknown R or reaction at the journal of the pulley The four action-lines are known, being vertical and tangent to the respective friction-circles in the manner shown. Note the direction of the uniform motion (to raise the load Q).

Again we employ § 329, regarding A as a horizontal *beam* or lever with supports in the verticals f and c, and loaded with S_1

and S_2, both known. We lay off $\overline{r\,..\,s} = S_1$, and $\overline{s\,..\,t} = S_2$, and take any pole O, drawing rays from O to r, s, and t. From any point c in the action-line of R (the *left-hand* reaction, or supporting force) we draw a line || to $O\,..\,r$ to find d, then $d\,..\,e$ || to $O\,..\,s$ to find e, and a line parallel to $O\,..\,t$ through e, to find f in the action-line of the *right-hand* supporting force, P. Drawing $c\,..\,f$, a line parallel to it through O fixes n' on the load-line (produced), giving $tn' = P$, and $n'\dot{w} = R$; i.e., $[A]$ is the force-diagram.

Without friction, the vertical action-lines would be drawn through the centres of the friction-circles, and a new construction on this basis would give P_0, whence the efficiency $P_0 \div P$ can be found.

For *backward motion* each force-vertical shifts over to the opposite side of the friction-circle from that shown in Fig. 17, and the result of a third construction is P'. If P' is found to be *negative*, that is, *if n' occurs above t* in diagram A, the mechanism is *self-locking*, as should be the case in the practical machine itself.

22. Rigidity of Hemp Ropes. Here, as with chains, the effect of the rigidity is to cause the tension where the rope is winding on to have a lever-arm about the centre of the pulley $= r + a$, where r = radius of circle formed by the axis of the rope when wound on the pulley, and a = a small distance which from Eytelwein's formula for rigidity of hemp ropes may be put $= 0.0093d^2$, where d is the diameter of the rope in *millimetres*, and whence a will be obtained in millimetres.

The tension on the unwinding side has a lever-arm of $r - a$. Hence, having computed a, we deal with hemp ropes as with chains. The phenomena observed with wire ropes are different. (See § 176.)

23. Tooth Friction in Spur Gearing. (Plate V, Fig. 18.) This figure shows one gear-wheel driving another, both provided with "involute teeth" by which we are to understand that the normal $a_1\,..\,o$, or $o\,..\,a_2$, at the point of contact always passes through o, the intersection of the line of centres with the pitch-circle, as motion proceeds.

We assume here that two pairs of teeth are always in contact.

Just now these points of contact are at a_1 and a_2, and have therefore a common normal a_1oa_2.

Rubbing occurs both at a_1 and a_2, and evidently in such directions that the pressure at a has $a_1\,b_1$ as action-line; and that at a_2 has b_2a_2 as action-line, making the angle of friction with the respective normals (or common normal, rather). This common normal a_1oa_2 will be assumed as making an angle of 75° with the line of centres at all times (property of the kind of teeth used). Hence the resultant action of the two driving teeth upon those driven is represented by an ideal force R, the resultant of the pressures at a_1 and a_2, and acting through o', making an angle of 75° with the line of centres. Notice the position of this angle with reference to the direction of motion and to the driven wheel; also that the effect of friction is to cause the action-line of R, which without friction would act along $a_1o\,a_2$, to be shifted $\|$ to itself a distance $\overline{oo'}$ farther from the centre of the *driving* wheel. This distance $\overline{oo'}$, can easily be determined by drawing the parts concerned on a convenient scale, and will be called ζ in the next paragraph.

24. Example XI. Pinion Spur-Wheel, Drum and Weight. (Plate V, Fig. 19.) The weight Q hangs by a chain or rope from the drum B which forms a rigid body with the spur-wheel H, with which the pinion A gears. A drives H, and it is required to find what force P, applied to a crank d (forming one piece with the pinion) and acting (at this instant) in the line $b\,..\,d$, will maintain uniform motion; i.e., overcome all frictions and raise Q without acceleration.

Since A drives H with tooth-gearing (involute and of same design as in preceding paragraph), the line of action of the resultant pressure R_2 between them is $bo'g$, making an angle of 75° with the line of centres, as shown (note on which side), and is drawn through the point o' on the line of centres but at a distance $= \zeta$ farther from the centre of the driving pinion A than a point in the pitch-circle of the latter. Call this force R_2.

The reaction at the bearing s is some force R_1 whose action-line must pass through b, the intersection of the action-lines of the other two forces, R_2 and P (since A is a three-force piece),

and be tangent (on the right) to the friction-circle at s. The action-line of Q is vertical, and is tangent (on the left) to a friction-circle at a (just as in Fig. 15 a similar relation holds at K); it cuts $g\,..\,b$ at g, and therefore a line drawn through g, and tangent (on right) to the friction circle at K, is the action-line of R_3, the reaction at the bearing K.

We thus have the action-lines of all three forces acting on each of the three-force pieces, A and H, while the force Q is given. Hence, the force-triangle $r\,..\,m\,..\,n$ is easily drawn for piece H, and determines R_2 and R_3. With $\overline{n''m''} =$ and $\parallel$ to $\overline{m\,..\,n}$ as a known side, we then complete the force-triangle for piece A, from which $\overline{m''\,..\,r''} = P$ is scaled off.

Without friction, R_2 would shift to the position $g_0\,..\,b_0$, R_1 would pass through b_0 and the *centre* of the circle at s. R_3 would pass through the *centre* of the circle at B and the point g_0, in the new vertical action-line of Q (through centre of friction-circle mentioned above). $g_0\,..\,b_0$ is parallel to $g\,..\,b$ and passes through the intersection o of the line of centres $c\,..\,s$, and the pitch-circle of the pinion A. Drawing the dotted force-triangles on this basis, Q being given, we finally obtain $P_0 = \overline{m''_0\,..\,r_0''}$. The efficiency can now be obtained, $= P_0 \div P$.

25. Belt Gearing. In Fig. 20, Plate VI, we have a pulley turning in a fixed bearing and driven by a force P. By belt connection this pulley drives another, not shown in the figure. The tension S_n, on the driving side is greater than that, S_0, on the following side. If Z is the (ideal) resultant of S_n and S_0, then the reaction R of the bearing must act in a line through b, the intersection of P and Z and tangent (above) to the friction-circle at the bearing; i.e., it acts along $a\,..\,b$. *If we assume that the belt is on the point of slipping* on the smaller of the two pulleys, we have the relation

$$S_n = S_0 e^{f\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\S\ 170)$$

where $f =$ coefficient of friction, e is the Naperian Base, and $\alpha =$ arc of contact on the smaller pulley in π measure, or in *radians*. Although S_n and S_0 are both unknown at the outset, we have

their ratio from the above equation, and hence can construct the action-line of Z (for *impending slip* only, it must be remembered), their resultant, thus determining the point b in Fig. 20 and ultimately R and Z, as will be seen. With Z found, we can obtain S_n and S_0. The value of this ratio, $e^{f\alpha}$, having been computed for a range of values of f and of α, the results may be embodied graphically in the spirals shown in Fig. 21, Plate VI, these being drawn in such a way, all starting from the point A in the circumference of the circle $A\,..\,C$, that if $\overline{OA}$, the radius, represent the smaller tension, S_0, and the special value $\alpha = AOC$ in any case be laid off and the radius OC produced till it intersects the spiral corresponding to the coefficient f proper to the case in hand, B being this intersection; then $\overline{OB} = S_n$, and $CB = S_n - S_0$ (which multiplied by the velocity of the belt gives the *power* transmitted). Or, whatever S_0 may be, the *ratio* $BO : AO =$ the *ratio* $S_n : S_0$, and may be obtained from the diagram if f and α are given. [N. B.—Note carefully that the relation $\frac{S_n}{S_0} = e^{f\alpha}$ only holds when the belt is *actually slipping on the pulley-rim* (and then f is the coefficient of friction of motion) or is *on the point of slipping* (and then $f =$ coefficient of friction of rest), and *is never to be used* except for those conditions. Of course, in most machinery impending slip is to be avoided, and the only use of the above formula in such cases is to find the ideal maximum value, $e^{f\alpha}$, for the *ratio* $S_n : S_0$, which the actual value should not approach if slipping is not to occur. For the uniform motion of an "idle pulley," ignoring axle friction, $S_n : S_0$ is always equal to 1.00.

. 26. **Example XII. Brake Strap and Drum.** (Plate VI, Fig. 22.) A is a lever with a fixed fulcrum or bearing at B and has attached to it both ends of the belt or strap which passes over a pulley and serves as a brake to prevent the acceleration of the descending weight Q. The chain sustaining Q unwinds from the drum C, rigidly attached to the pulley, which turns on a fixed bearing B. Required the proper force P, in a given action-line $r\,..\,P$, to be applied to the lever A, to preserve a uniform motion for Q (downward).

Evidently, from the direction of motion, the tension in the strap at d is the greater, $= S_n$, and that in the portion $x..e$ is the smaller, $= S_0$. Since in this case there is actual slipping of the drum under the strap, the ratio $S_n : S_0$ is known from Fig. 21, for (say) $f = 0.18$ and $\alpha = \frac{3}{2}\pi$ (corresponding to 270°), to be 2.51. Hence, from the intersection, e, of the two straight portions of the belt, we lay off any convenient distances $a..e$ and $a..d$, such that $\overline{ad} = 2.51\,\overline{ae}$, and complete a parallelogram upon them as sides; then the diagonal $a..o$ is the action-line of Z, the unknown resultant of S_n and S_0.

The pulley and drum, therefore, constitute a three-force piece acted on by Q in the vertical line tangent (on the inside) to a friction-circle at t; by the (ideal) Z in line $a..o$; and by the reaction, R, of the bearing B, tangent (on left) to the friction-circle and in a line which must pass through g' (intersection of the other two force-lines). Knowing Q and all three action-lines mentioned, the force-triangle $k..m..n$ determines R and Z; i.e., $\overline{n..k}$ and $\overline{m..n}$. As for the lever A, though there are really four forces acting on it, viz., P, S_n, S_0, and R_1 (the reaction at B), we may treat it as a three-force piece, since S_n and S_0 are equivalent to the force Z, now known both in amount and position. Hence we draw $h..f =$ and || to Z, then $f..g$ || to $r..s$ (since Z and P meet at r) and $h..g$ || to $r..P$, thus fixing $R_1 = \overline{f..g}$ || and $P = \overline{g..h}$. Then by resolving Z along $a..d$ and $a..e$ we find S_n and S_0. (*Note.*—Q is the *working force* in this example, and P *neutral*. All work is spent in friction.)

27. Example XIII. Transmission of Power through two Pulleys having Belt Connection. (Plate VI. Fig. 23.) Let the pulley A drive the pulley B with uniform motion. At this instant the useful resistance is Q acting on B in line $m..q$. Given Q, required P, acting in line $b..a$ on pulley A, to overcome Q and the journal frictions at B and B'. We assume that the belt is on *the point of slipping* on the circumference of the *smaller* pulley, B; then $S_n = S_0\, e^{f\alpha}$, where α is the arc of contact on the *small* pulley. Prolong the action-lines of S_n and S_0 to their intersection o. From Fig. 22 find the ratio of S_n to S_0

for the given coefficient of friction and the arc α, and lay off $o \, . . \, l$ and $o \, . . \, k$ in this ratio ($\overline{ok} > \overline{ol}$) at convenience. This gives $o \, . . \, i$ as the action-line of Z, the (ideal) resultant of S_n and S_0; of course, that there may be no slip or any approach to it, actual values must be secured for these tensions *greater* than those to be found by this construction, which are for *impending* slip on small pulley (and this means the assumption of a *less* value for the ratio $S_n : S_0$. (See p. 186.)

A is a three-force piece (so considered here) under the action of Z (ideal), P, and R_1 the bearing reaction. Pulley B may also be treated as a three-force piece under action of Q, of Z (reversed) and R, the bearing reaction at B. P and Z intersect at b, Q and Z at q. Hence R_1 acts through b and tangent (above) to its friction-circle; while R acts through q and is tangent (on right) to friction-circle at B.

Beginning, then, with pulley B, since the force Q is given, we close the triangle $v \, . . \, w \, . . \, x$ in an obvious manner, obtaining R and Z. For pulley A, now that Z is found, we complete the force-triangle $y \, . . \, z \, . . \, d$, and determine R_1 and P.

Without friction at the *bearings*, R and R_1 would pass through the centres of their friction-circles and the dotted force-triangles would result, whence we have $P_0 = \overline{z \, . . \, d'}$.

To find the belt tensions (for impending slip on smaller pulley) we resolve the force $Z \parallel$ to their directions; see lower part of the figure.

Note.—As far as finding the value of P alone is concerned, having Q given, *any line whatever* could be taken through the point o, as the action-line of the resultant of the two tensions, if the friction at the bearings were disregarded, and the construction would result in the same value of P, whatever the belt-tensions, provided the belt did not slip.

28. Final Remark. From an inspection of the preceding examples involving the effect of friction in the working of machines, it becomes apparent, as should be expected, of course, that in every case this effect is to put the working force at the *greatest possible disadvantage*, thus exacting as large a value as possible for it; and from this general principle we may often decide quickly in the matter of tangencies to friction-circles, inclination of a pressure on one side or the other from the normal, etc.

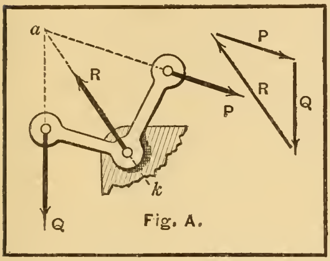

Fig. A.

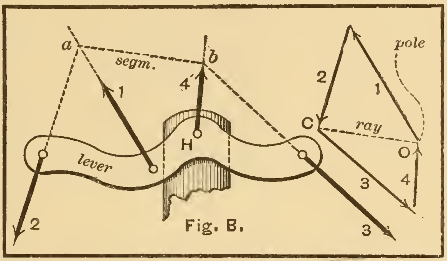

Fig. B.

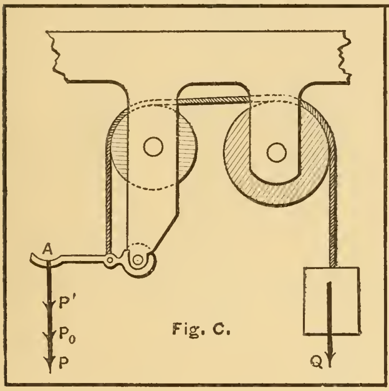

Fig. C.

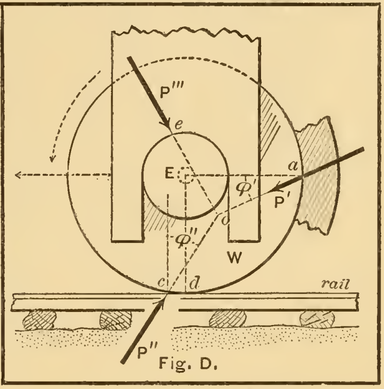

Fig. D.

Am. Bk. Note Co. N. Y.

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PLATE I. Figures 1 to 4.

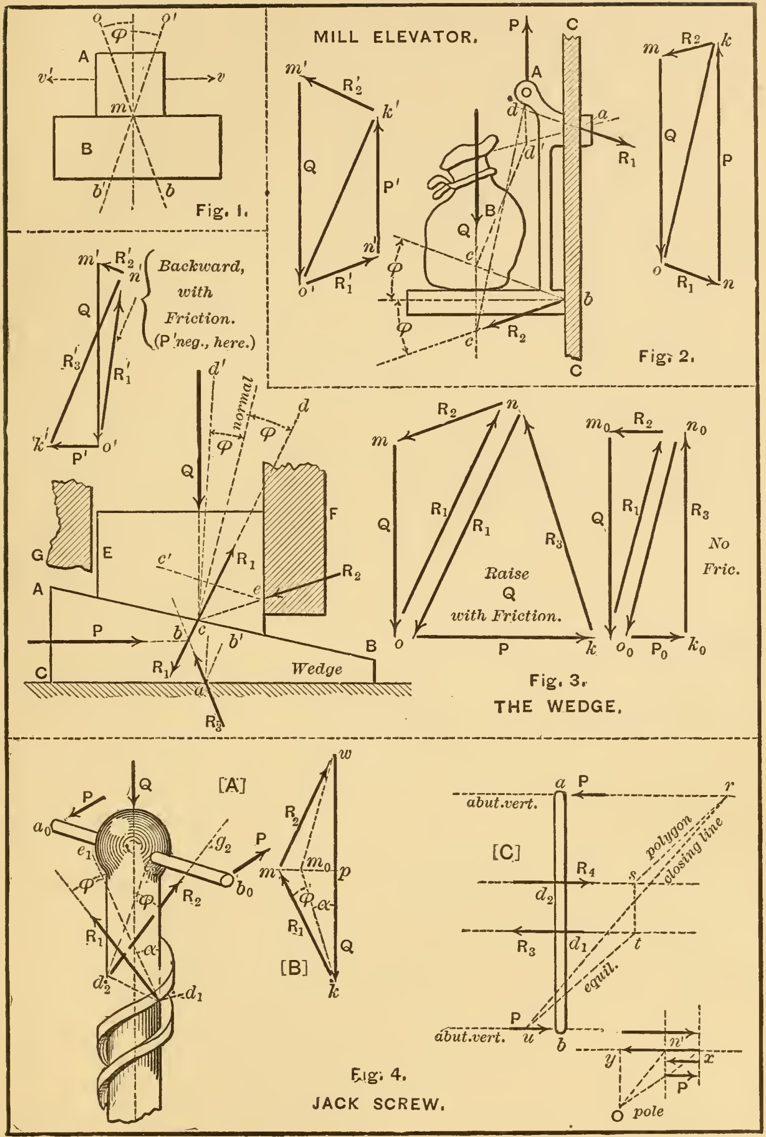


Fig. 1.

MILL ELEVATOR. Fig. 2.

Fig. 3. THE WEDGE.

Fig. 4. JACK SCREW.

Am. Bk. Note Co. N. Y.

PLATE II. Figures 5 to 8

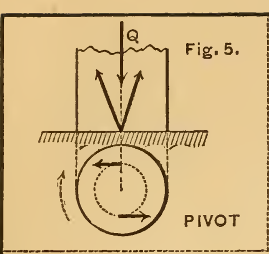


Fig. 5.

PIVOT

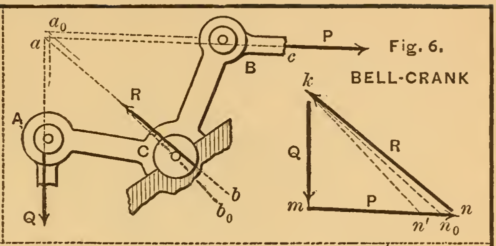


Fig. 6.

BELL-CRANK

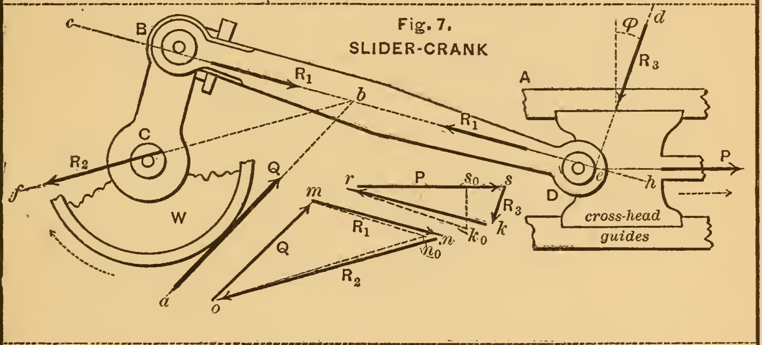


Fig. 7.

SLIDER-CRANK

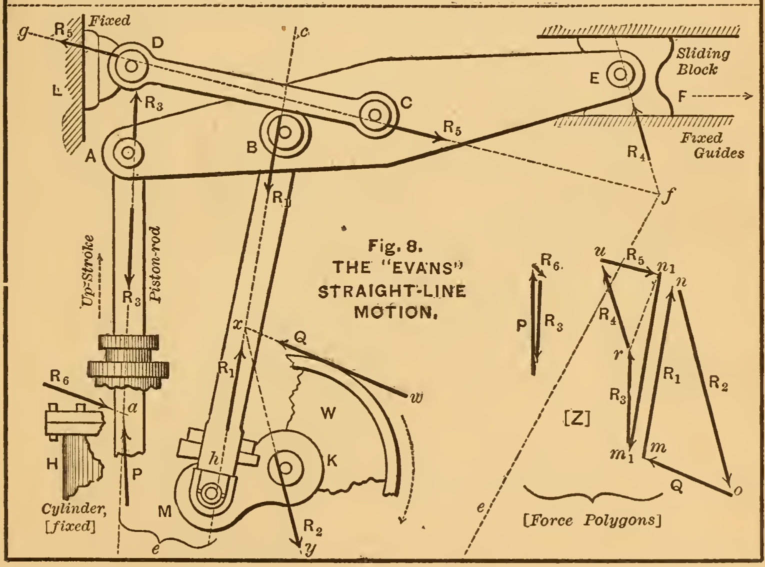
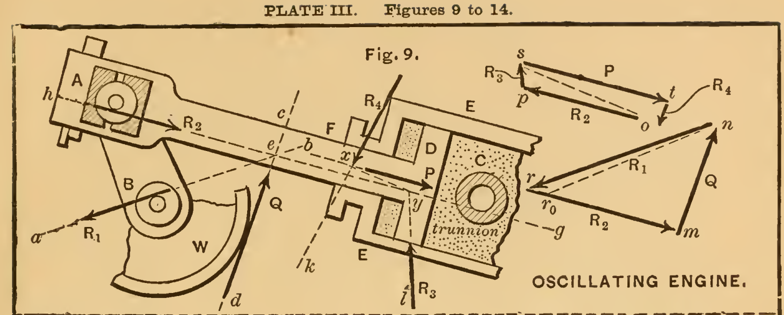


Fig. 8.

THE "EVANS" STRAIGHT-LINE MOTION.

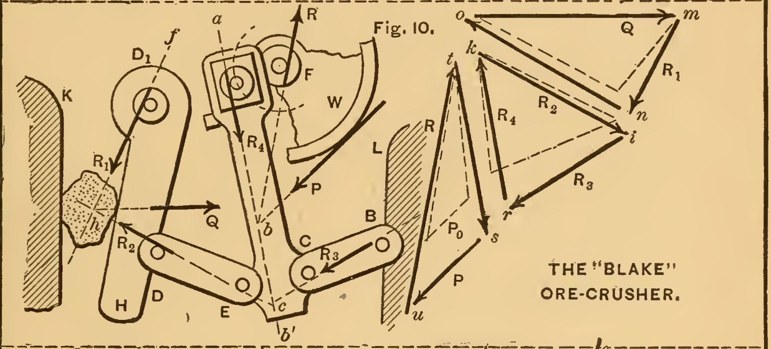
PLATE III. Figures 9 to 14.

Fig. 9.



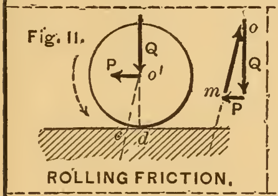
OSCILLATING ENGINE.

Fig. 10.



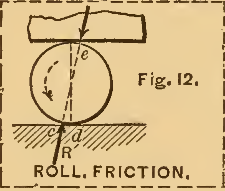
THE "BLAKE" ORE-CRUSHER.

Fig. 11.



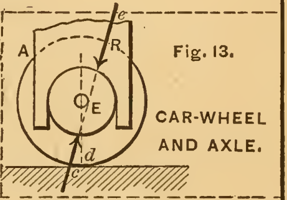
ROLLING FRICTION.

Fig. 12.



ROLL. FRICTION.

Fig. 13.



CAR-WHEEL AND AXLE.

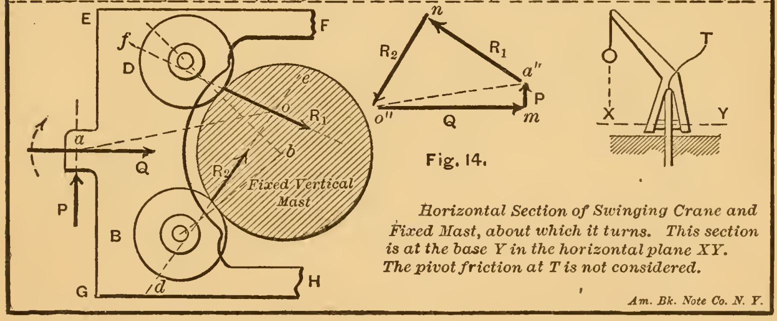
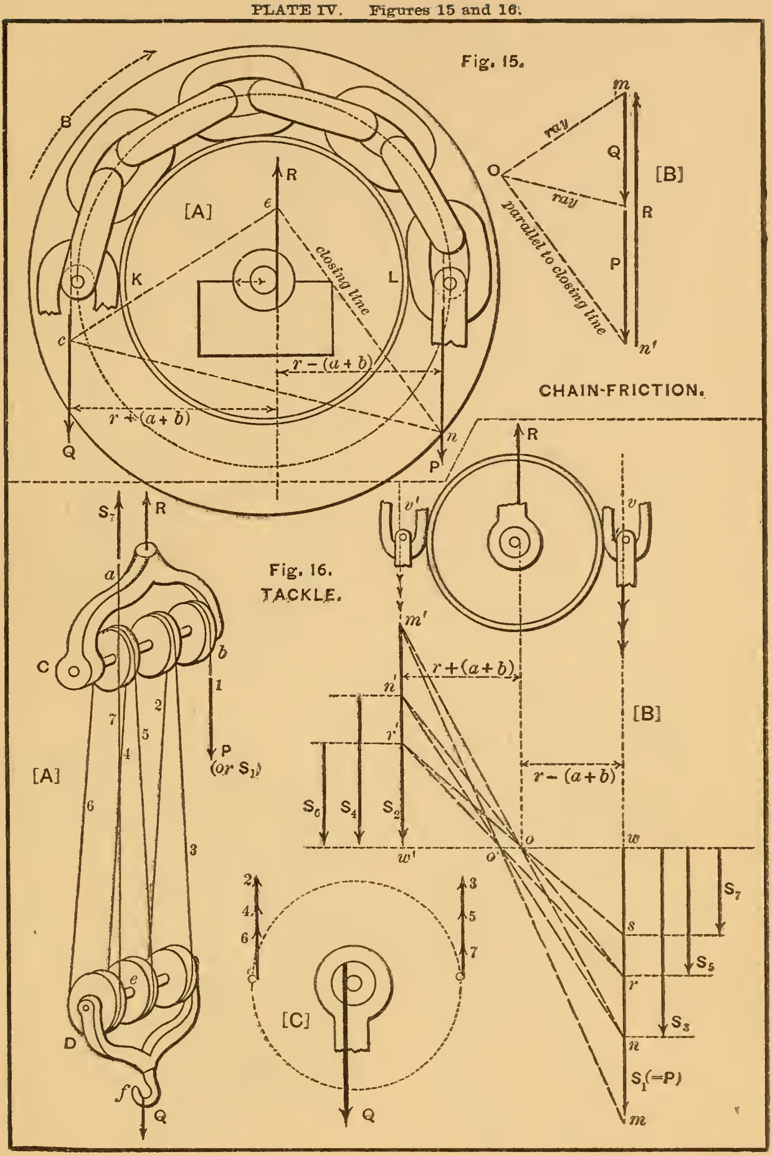


Fig. 14.

Horizontal Section of Swinging Crane and Fixed Mast, about which it turns. This section is at the base Y in the horizontal plane XY. The pivot friction at T is not considered.

Am. Bk. Note Co. N. Y.

PLATE IV. Figures 15 and 16.



Am. Bk. Note Co. N. Y.

PLATE V. Figures 17 to 19.

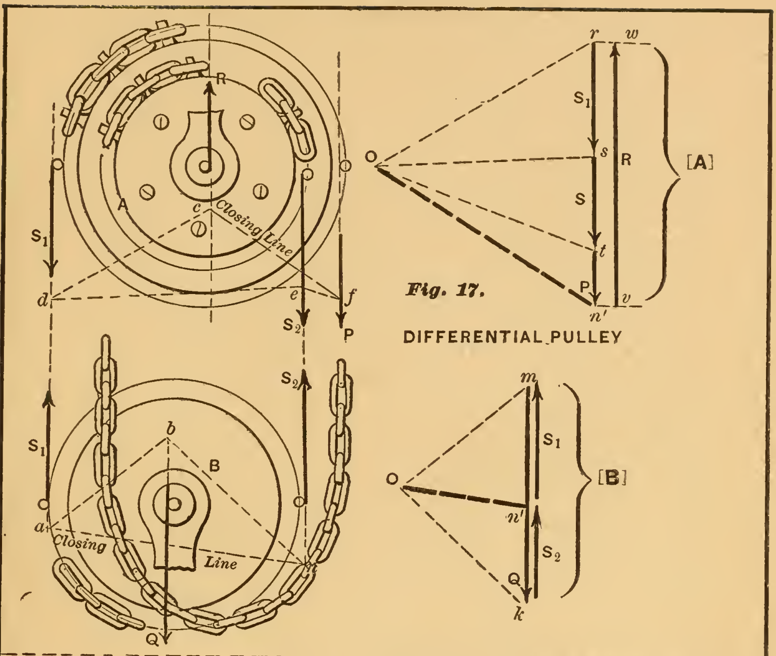


Fig. 17.

DIFFERENTIAL PULLEY

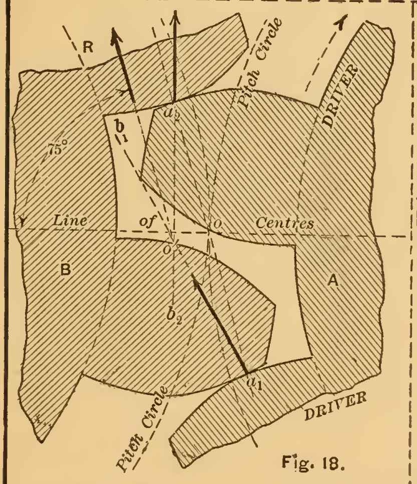


Fig. 18.

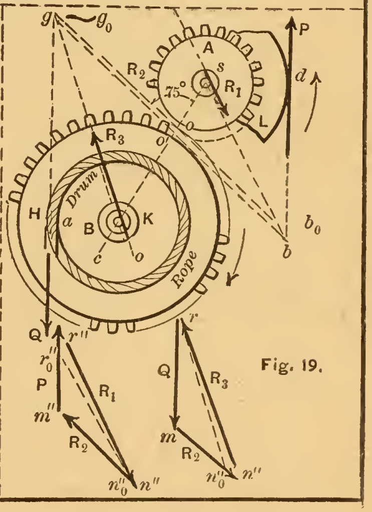


Fig. 19.

PLATE VI. Figures 20 to 23.

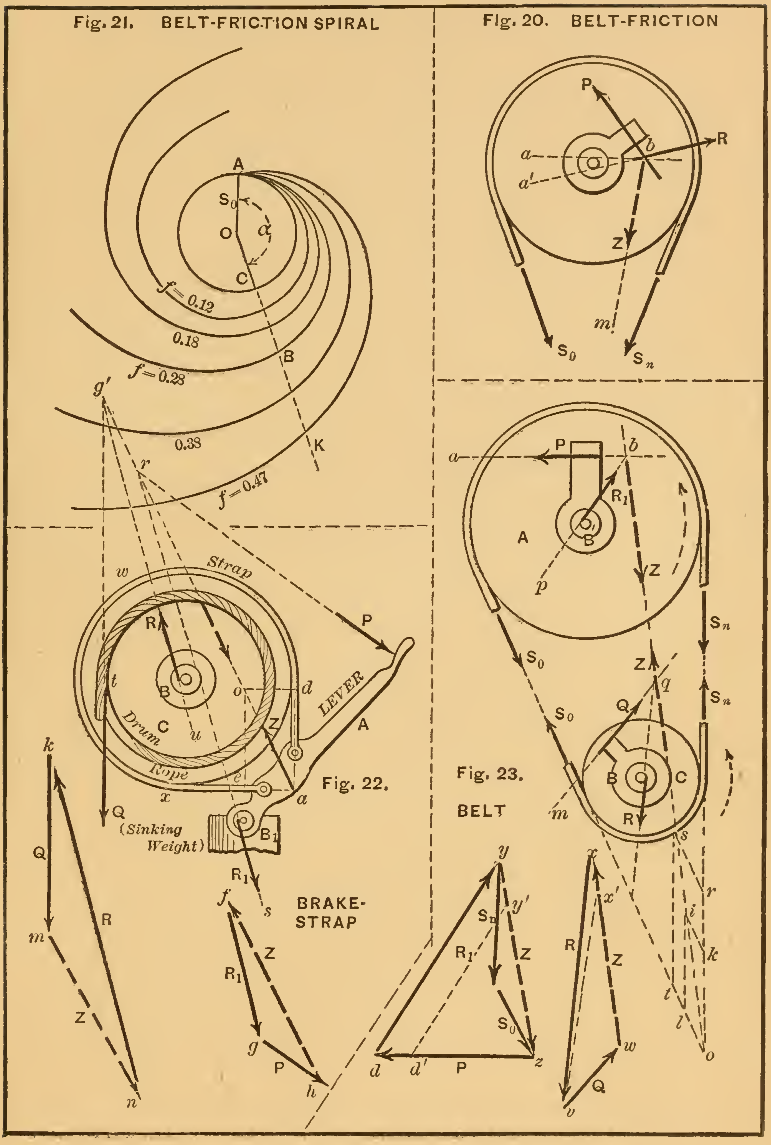


Fig. 21. BELT-FRICTION SPIRAL

Fig. 20. BELT-FRICTION

Fig. 22. BRAKE-STRAP

Fig. 23. BELT

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